Learning Control of Linear Iteration Varying Systems with Varying References through Robust Invariant Update Laws*

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***Abstract*— Iterative learning control (ILC) has long been recognized as an efficient way of improving the tracking performance of repetitive systems. While ILC can offer significant improvement to the transient response of complex dynamical systems, the fundamental assumption of iteration invariance of the process limits potential applications. Utilizing abstract Banach spaces as our problem setting, we develop a general approach that is applicable to the various frameworks encountered in ILC. Our main result is that robust invariant update laws lead to stable behavior in ILC systems, where iteration varying systems converge to bounded neighborhoods of their nominal counterparts when uncertainties are bounded. Furthermore, if the uncertainties are convergent along the iteration axis, convergence to the nominal case can be guaranteed.**

I. INTRODUCTION

Ever since Arimoto's 1984 paper [1], iterative learning control (ILC) has been recognized as an efficient way of improving the tracking performance of repetitive systems. ILC can offer significant improvement to the transient response of complex dynamical systems with a high level of uncertainty through very simple algorithms [2], [3]. Traditionally, the fundamental assumption of ILC has been iteration invariance of the: 1) plant dynamics, 2) exogenous disturbances, 3) initial conditions, and 4) reference. This assumption greatly simplifies the ILC problem and enables the control engineer to design a stable recurrence in the iteration domain by employing a contraction mapping. Even though this assumption is highly unrealistic, similar to feedback control of linear time invariant systems, it yields good results in practice provided that the variation of the process parameters from trial to trial is small.

Nevertheless, to realize the full potential of "learning" control, it is imperative that the fundamental invariance assumption is relaxed: In practice, initial conditions and disturbances are always subject to variations, while references and plants can commonly appear as outputs of higher order internal models (HOIMs) in the context of robotic manipulators doing different tasks, or freeway traffic models [4]. One of the earliest efforts to analyze iteration varying systems is presented in [5], although the analysis is far from complete. Since then, several other works have tackled the case of iteration varying references [6] and noise [7]. Of particular mention is [8], where the authors have developed a HOIM based learning law using the w transform introduced in [9] to tackle iteration varying references and plants.

Recently, the robust ILC problem for linear discrete time (LDT) systems with uncertain state matrices was considered in [10]. The authors show in [10] that if a time and iteration varying learning matrix can be designed to ensure contractions over finite iteration intervals, the system converges to a neighborhood of 0 provided the state matrices are bounded. While the results of this paper are theoretically important, the analysis is constrained to discrete time systems in state space form. Additionally, the authors make no comments on how this learning matrix can be designed when the sole information on the uncertainty is boundedness. With these issues in mind, in this paper we consider the robust ILC problem of iteration varying linear systems in an abstract setting. Our main result is that if the uncertainties are within a bounded neighborhood of a nominal system, the system signals converge to a bounded neighborhood of their nominal counterparts as long as the first order update law is robustly monotonically convergent over all uncertain plants, which can be designed using existing methods from the literature.

The remainder of the manuscript is organized as follows: Section II introduces certain preliminaries and the ILC problem. Section III proves the basic boundedness result of the algorithm. Sec IV introduces a nominal system to facilitate further analysis and shows that the choice of initial input does not affect the system in the limit. Section V compares the limiting performance of the algorithm to the nominal case; the effect of the design choices on the limiting performance is studied in section VI. Section VII discusses how the so called Q filter in linear ILC affects system performance in the iteration invariant and varying cases. Finally, concluding remarks are given in section VIII.

II. PROBLEM FORMULATION

Let $H : \mathcal{U} \to \mathcal{Y}$ be a mapping where $\mathcal{U}$ is the space of admissible inputs and $\mathcal{Y}$ is the space of outputs. Assuming that H is known and there are no exogenous inputs affecting the output, the classical ILC problem can be stated as that of finding a controller C that maps the input history $u_0, u_1, \ldots, u_{k-1} \in \mathcal{U}$ to the current input $u_k \in \mathcal{U}$ such that the output $y_k = Hu_k$ converges to a desired reference r in the image of H, or a small neighborhood of it, as $k \to \infty$. In most cases, C is designed to consider only the previous iteration, thus giving rise to the name first order ILC.

For our problem, we will assume $\mathcal{U}$ and $\mathcal{Y}$ to be Banach spaces equipped with suitable norms. We base this assumption on the fact that Banach spaces are the natural

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settings of contraction mapping based ILC, which relies on the celebrated Banach fixed point theorem. Furthermore, $\mathcal{L}_p$ and l_p spaces, which provide the standard framework in time driven dynamic systems, are well known to be complete normed spaces. The motivation for this assumption is to come up with a general framework that contains the variety of different settings in ILC, consistent with the approach in [3].

A. Notation and Preliminaries

We take $\mathbb{N}$ to represent the set of nonnegative integers. We denote by $\mathcal{B}(\mathcal{U},\mathcal{Y})$ the space of all bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$. We use $\|.\|$ to denote vector and induced operator norms in the relevant spaces. For a family of operators indexed by a subset of $\mathbb{N}$, we use the product notation to indicate the composition of the operators in increasing order; e.g. $\prod_{i=j}^{k} H_i \triangleq H_k H_{k-1} \dots H_j$ for $j \leq k$ and $\prod_{i=j}^{k} H_i \triangleq I$ for $j > k$, where I is the identity operator.

For a rigorous study of the convergence and stability of the iterative problem, we define the spaces $\mathcal{U}^\omega \triangleq \prod_{k\in\mathbb{N}} \mathcal{U}$ and $\mathcal{Y}^\omega \triangleq \prod_{k\in\mathbb{N}} \mathcal{Y}$. An element x in these spaces will be defined so that x_k denotes the kth coordinate. Alternatively, we define x to be a mapping from $\mathbb{N}$ to $\mathcal{U}$ or $\mathcal{Y}$. Throughout the rest of the manuscript we will use this notation to refer to any sequence of objects in the same space, e.g. $x \triangleq (x_0, x_1, \dots)$ where each x_k can be an element of $\mathcal{U}$, $\mathcal{Y}$, or an operator in these spaces. In addition, we introduce the following, where the spaces $\mathcal{X}$ and $\mathcal{V}$ are in $\{\mathcal{U},\mathcal{Y}\}$.

Definition 1: Let x be an element of $\mathcal{X}^\omega$. The norm of x is given by $\|x\| \triangleq \sup_{k\in\mathbb{N}} \|x_k\|$.

Definition 2: A linear mapping $H : \mathcal{X}^\omega \to \mathcal{V}^\omega$ is called *bounded-input bounded-output (BIBO) stable* if there exists a finite constant α such that

$$\|(Hx)_\kappa\| \leq \alpha \|(x)_\kappa\|, \quad \forall x \in \mathcal{X}^\omega, \forall \kappa \in \mathbb{N},$$

where $(x)_\kappa \triangleq (x_0, x_1, \dots, x_\kappa, 0, 0, \dots)$ is the truncation of x.

Readers should note that $\mathcal{U}^\omega$ and $\mathcal{Y}^\omega$ are not normed spaces since our definition of the norm entails the possibility of unbounded elements. However, this is merely a formality and will not affect our analysis as any truncated vector in these spaces has a finite norm. This is akin to the definition of input-output stability via the extended space $\mathcal{L}_{pe}$.

Definition 3: Let $x, v \in \mathcal{X}^\omega$. We say that *$x$ converges to v* if $\lim_{k\to\infty} \|x_k - v_k\|$ equals 0. If $\limsup_{k\to\infty} \|x_k - v_k\|$ is bounded, we say *x converges to a bounded neighborhood of v*.

Definition 4: Let $H_k \in \mathcal{B}(\mathcal{X},\mathcal{X})$. An iterative system defined by the equality $x_{k+1} = H_k x_k$ for all $k \in \mathbb{N}$ is *asymptotically stable* if there exists α such that $\|x\| \leq \alpha \|x_0\|$ and x converges to 0 for all $x_0 \in \mathcal{X}$.

B. System Dynamics

Based on the above, we consider the following class of systems

$$y_k = P_k u_k + d_k, \quad \forall k \in \mathbb{N}, \tag{1}$$

where $y_k \in \mathcal{Y}$ is the output, $u_k \in \mathcal{U}$ is the input, $d_k \in \mathcal{Y}$ is the exogenous signal that includes the feedback control response, disturbance, noise, and the effect of initial conditions, and P_k is the iteration varying linear input-output operator. Moreover, we assume that each P_k is in the vicinity of a known bounded linear operator $\bar{P}$ as stated in the following:

Assumption 1: The input-output operators lie in a neighborhood of $\bar{P}$. In other words, there exists a finite real constant ρ such that

$$P_k \in \mathcal{P} \triangleq \left\{ H \in \mathcal{B}(\mathcal{U},\mathcal{Y}) : \|H - \bar{P}\| < \rho \right\}, \quad \forall k \in \mathbb{N}.$$

The model we have described is fairly general and our only limiting assumption is linearity. For instance, if we assume a time dependent dynamical system, P_k can be time invariant or varying. Input disturbances can be assumed to be included in d_k when mapped by P_k. Due to the assumption that the process variables P_k and d_k are varying along the iteration axis, it is a straightforward matter to assume that the reference is also subject to variations from trial to trial. Thus, our objective is to find an ILC update law such that the error vector e defined by $e_k \triangleq r_k - y_k$ for all $k \in \mathbb{N}$, where the reference r_k is in the image of $\bar{P}$ for all $k \in \mathbb{N}$, converges to a small neighborhood of 0. As with the plant operators, we make a boundedness assumption on r.

Assumption 2: The reference vectors lie in a neighborhood of a nominal reference $\bar{r}$ in the image of $\bar{P}$. In other words, there exists a finite real constant ζ such that

$$r_k \in \mathcal{R} \triangleq \left\{ h \in \bar{P}(\mathcal{U}) \subset \mathcal{Y} : \|h - \bar{r}\| < \zeta \right\}, \quad \forall k \in \mathbb{N}.$$

III. ROBUST INVARIANT UPDATE LAWS AND STABILITY

Let

$$u_{k+1} = Q u_k + L e_k, \quad \forall k \in \mathbb{N}, \tag{2}$$

where Q and L are bounded linear operators, and u_0 is arbitrary. Furthermore, the update law is subject to the robustness assumption that guarantees monotonic convergence in $\mathcal{P}$ when the system is iteration invariant, as stated below.

Assumption 3: There exists a real constant γ such that

$$\|Q - LH\| \leq \gamma < 1, \quad \forall H \in \mathcal{P}.$$

Substituting the system dynamics (1) into the update law (2) yields the recurrence relation

$$u_{k+1} = T_k u_k + L \eta_k, \quad k \in \mathbb{N},$$

where $T_k \triangleq Q - L P_k$ and $\eta_k \triangleq r_k - d_k$. The solution of the input vector in terms of u_0 and η_k can then be given as

$$u_{k+1} = \left(\prod_{i=0}^{k} T_i \right) u_0 + \sum_{i=0}^{k} \left(\prod_{j=i+1}^{k} T_j \right) L \eta_i, \tag{3}$$

for all $k \in \mathbb{N}$.

We are now ready to prove our first results, which say that the recurrence relation on the input vector is asymptotically and BIBO stable.

Proposition 1: The linear iterative system (3) with $\eta = 0$ is asymptotically stable.

Proof: Take arbitrary $u_0 \in \mathcal{U}$. Then from (3) we have $\|u_{k+1}\| \leq \gamma^{k+1}\|u_0\|$. Since $\gamma < 1$, it follows that u converges to 0 and $\|u\| \leq \|u_0\|$. Therefore, system (3) is asymptotically stable. ■

Proposition 2: The linear iterative system (3) with input η and $u_0 = 0$ is BIBO stable.

Proof: Take arbitrary $\eta \in \mathcal{Y}^\omega$. Then from (3) we have

$$\begin{aligned}\|u_{\kappa+1}\| &\leq \sum_{i=0}^{\kappa} \gamma^{\kappa-i}\|L\|\|(\eta)_\kappa\| = \frac{1-\gamma^{\kappa+1}}{1-\gamma}\|L\|\|(\eta)_\kappa\| \\ &\leq \frac{\|L\|\|(\eta)_\kappa\|}{1-\gamma} \leq \frac{\|L\|\|(\eta)_{\kappa+1}\|}{1-\gamma}, \quad \forall \kappa \in \mathbb{N}, \end{aligned} \tag{4}$$

where we use the fact that the truncated norm of η is monotonically increasing by definition. Using the same property, we can show by (4) that $\|(u)_\kappa\| \leq \|L\|\|(\eta)_\kappa\|/(1-\gamma)$ for all $\kappa \in \mathbb{N}$. Therefore, system (3) is BIBO stable. ■

We showed that the iterative system (3) is asymptotically and BIBO stable when the update law (2) is subject to the robustness assumption 3. We finish this section with the following theorem which show that u and y are bounded if d is bounded.

Theorem 1: The signals u and y of the linear iterative system 1 with the update law (2) is bounded if d is bounded.

Proof: Consider the solution (3) of the input u, which is the superposition of the *natural response* describing the asymptotic response to the initial condition u_0 and the *forced response* describing the input-output behavior due to η. Since r is bounded by assumption 2, η is bounded if d is bounded. From propositions 1 and 2, it follows that u is bounded. Now observe that

$$\|y_k\| \leq \|P_k\|\|u_k\| + \|d_k\| \leq \|P_k\|\|u\| + \|d\|, \quad \forall k \in \mathbb{N},$$

by (1). Since $\mathcal{P}$ is uniformly bounded, it follows that y is bounded. ■

IV. ASYMPTOTIC SYSTEM DYNAMICS

In the previous section we showed that the ILC system is well posed when subject to the robustness assumption, in the sense that bounded inputs produce bounded outputs for any initial input. Furthermore, we saw that LDT methods apply directly to the iteration varying system regardless of what $\mathcal{U}$ and $\mathcal{Y}$ are. While there are more general conditions in finite dimensional spaces, e.g. the joint spectral radius being less than 1, the robustness condition of assumption 3 will suffice for our case since it guarantees monotonic convergence for iteration invariant systems.

We note that while boundedness of the signals are sufficient to show that they converge to a bounded neighborhood of 0, we would like to find tighter bounds if possible. In particular, we would like to see how the performance is affected when compared to iteration invariant systems. One motivation for analyzing these systems in general as opposed to systems that only converge to the origin is that perfect tracking can be an infeasible objective for various reasons. Hence, in this section, we show that the dynamics of the ILC system can be approximated asymptotically by an auxiliary system. Furthermore, we introduce a nominal iterative system via the known operator $\bar{P}$ and reference $\bar{r}$ under the assumption that $d = 0$, and show that this system can also be approximated by a nominal auxiliary system. These will later facilitate our analysis of asymptotic performance and will describe the "steady state" behavior of the systems after the effects of the initial inputs vanish.

A. Actual System

The error dynamics of the actual system is given by the relation

$$e_k = -P_k u_k + \eta_k, \quad \forall k \in \mathbb{N}.$$

Further, we introduce the auxiliary variables x and z

$$\begin{aligned} x_{k+1} &\triangleq \textstyle\sum_{i=0}^{k} \left(\prod_{j=i+1}^{k} T_j\right) L\eta_i, \\ z_k &\triangleq -P_k x_k + \eta_k, \end{aligned} \tag{5}$$

for all $k \in \mathbb{N}$, where x_0 is arbitrary. The auxiliary variables define the asymptotic behavior of the system as shown in the following results.

Proposition 3: The input u and the error e of the linear iterative system described by (1) with the update law (2) converge to x and z, respectively.

Proof: Subtracting x from u yields the natural response of the input dynamics which converges to 0 since it is asymptotically stable by proposition 1. Hence, u converges to x. Similarly, subtracting z from e yields

$$e_k - z_k = -P_k(u_k - x_k), \quad \forall k \in \mathbb{N},$$

which tends to 0 since $\mathcal{P}$ is uniformly bounded and u converges to x. Therefore, e converges to z. ■

B. Nominal System

We define the nominal system to be the case where the signal $d = 0$ and the plant $P_k = \bar{P}$ for all $k \in \mathbb{N}$. In other words, we describe the nominal system as

$$\bar{y}_k = \bar{P}\bar{u}_k, \quad \forall k \in \mathbb{N},$$

where $\bar{y}_k \in \mathcal{Y}$ is the nominal output and $\bar{u}_k \in \mathcal{U}$ is the nominal input. Thus, the error dynamics of the nominal system is given by the relation below, where $\bar{\eta} \triangleq \bar{r}$:

$$\bar{e}_k = -\bar{P}\bar{u}_k + \bar{\eta}, \quad \forall k \in \mathbb{N}.$$

We take the update law as $\bar{u}_{k+1} = Q\bar{u}_k + L\bar{e}_k$ with Q and L the same as before. Further we introduce the nominal auxiliary variables $\bar{x}$ and $\bar{z}$,

$$\begin{aligned} \bar{x}_{k+1} &\triangleq \textstyle\sum_{i=0}^{k} \left(\prod_{j=i+1}^{k} \bar{T}\right) L\bar{\eta}, \\ \bar{z}_k &\triangleq -\bar{P}\bar{x}_k + \bar{\eta}, \end{aligned} \tag{6}$$

for all $k \in \mathbb{N}$, where $\bar{T} \triangleq Q - L\bar{P}$ and $\bar{x}_0$ is arbitrary. Similar to the actual system, the signals define the asymptotic behavior as given by the following result stated without proof.

Proposition 4: The nominal input $\bar{u}$ and error $\bar{e}$ converge to $\bar{x}$ and $\bar{z}$, respectively.

V. ASYMPTOTIC LEARNING PERFORMANCE

We will now analyze the performance of the algorithm (2) on the ILC system. Towards that end, based on the results of the previous section, we will compare the auxiliary systems rewritten below in recursive form:

$$\bar{x}_{k+1} = \bar{T}\bar{x}_k + L\bar{\eta}, \quad \forall k \in \mathbb{N}, \tag{7}$$

$$x_{k+1} = T_k x_k + L\eta_k, \quad \forall k \in \mathbb{N}, \tag{8}$$

where $\bar{x}_0 = x_0 = 0$. Furthermore we let

$$\begin{aligned} &\tilde{x}_k \triangleq \bar{x}_k - x_k, \quad \tilde{\eta}_k \triangleq \bar{\eta} - \eta_k, \quad \tilde{r} \triangleq \bar{r} - r_k, \\ &\tilde{T}_k \triangleq \bar{T} - T_k, \quad \tilde{P}_k \triangleq \bar{P} - P_k, \end{aligned}$$

for all $k \in \mathbb{N}$. We are now ready to show that the iteration varying ILC system converges to a bounded neighborhood of the nominal invariant system.

Theorem 2: Given a linear iterative system described by (1) with the update law (2), if d is bounded, u and e converge to a neighborhood of $\bar{u}$ and $\bar{e}$, respectively.

Proof: By propositions 3 and 4, we know that u and $\bar{u}$ converge to x and $\bar{x}$, respectively. Hence, it suffices to prove that x converges to a bounded neighborhood of $\bar{x}$. Observe that $\tilde{T}_k = (Q - L\bar{P}) - (Q - LP_k) = -L\tilde{P}_k$ and $\tilde{\eta}_k = \bar{r} - (r_k - d_k) = \tilde{r}_k + d_k$, so by subtracting (8) from (7) we arrive at

$$\tilde{x}_{k+1} = T_k\tilde{x}_k - L\tilde{P}_k\bar{x}_k + L(\tilde{r}_k + d_k), \quad \forall k \in \mathbb{N}.$$

By assumption 3

$$\|\tilde{x}_{k+1}\| \le \gamma\|\tilde{x}_k\| + \|L\|(\|\tilde{P}_k\|\|\bar{x}_k\| + \|\tilde{r}_k\| + \|d_k\|), \tag{9}$$

for all $k \in \mathbb{N}$. Now recall that $\bar{x}$ represents the forced response of the nominal input $\bar{u}$, which converges to a fixed point $\bar{x}_\infty$. Hence

$$\limsup_{k\to\infty} \|\tilde{x}_k\| \le \gamma \limsup_{k\to\infty} \|\tilde{x}_k\| + \|L\|(\rho\|\bar{x}_\infty\| + \zeta + \|d\|),$$

for all $k \in \mathbb{N}$. It follows that

$$\limsup_{k\to\infty} \|\tilde{x}_k\| \le \|L\| \frac{\rho\|\bar{x}_\infty\| + \zeta + \|d\|}{1-\gamma}. \tag{10}$$

Therefore, u converges to a bounded neighborhood of $\bar{u}$. Similarly, by propositions 3 and 4, we know that e and $\bar{e}$ converge to z and $\bar{z}$, respectively. Thus, it will suffice to prove that z converges to a bounded neighborhood of $\bar{z}$. Let $\tilde{z}_k \triangleq \bar{z}_k - z_k$ for all $k \in \mathbb{N}$. Subtracting z from $\bar{z}$ by (5) and (6) we have

$$\tilde{z}_k = P_k\tilde{x}_k - \tilde{P}_k\bar{x}_k + \tilde{r}_k + d_k, \quad \forall k \in \mathbb{N},$$

which leads to the inequality

$$\|\tilde{z}_k\| \le (\|\bar{P}\| + \|\tilde{P}_k\|)\|\tilde{x}_k\| + \|\tilde{P}_k\|\|\bar{x}_k\| + \|\tilde{r}_k\| + \|d_k\|, \tag{11}$$

for all $k \in \mathbb{N}$. From above, by substituting (10) it follows that

$$\begin{aligned} \limsup_{k\to\infty} \|\tilde{z}_k\| \le &\left(\|L\| \frac{\|\bar{P}\| + \rho}{1-\gamma} + 1\right) \\ &\times (\rho\|\bar{x}_\infty\| + \zeta + \|d\|). \end{aligned} \tag{12}$$

Therefore, e converges to a bounded neighborhood of $\bar{e}$. ■

Remark 1: Readers should note from the bounds in (10) and (12) that the system converges to the nominal case when the disturbance vanishes, and the uncertainty in the plant and the reference tend to 0.

Furthermore, we show that if the input-output operator and the reference converge to the nominal case, and d converges to 0, the ILC system converges to the nominal invariant system. In the following theorem, convergence of P to $\bar{P}$ is to be interpreted as $\lim_{k\to\infty} \|P_k - \bar{P}\| = 0$ as in definition 3.

Theorem 3: Given a linear iterative system described by (1) with the update law (2), if P converges to $\bar{P}$, r converges to $\bar{r}$, and d converges to 0, u and e converge to $\bar{u}$ and $\bar{e}$, respectively.

Proof: Consider (9). Then we have

$$\begin{aligned} \limsup_{k\to\infty} \|\tilde{x}_k\| \le &\gamma \limsup_{k\to\infty} \|\tilde{x}_k\| \\ &+ \|L\| \limsup_{k\to\infty} \left(\|\tilde{P}_k\|\|\bar{x}_k\| + \|\tilde{r}_k\| + \|d_k\|\right), \end{aligned}$$

which by the convergence assumptions on P, r and d implies $\limsup_{k\to\infty} \|\tilde{x}_k\| \le \gamma \limsup_{k\to\infty} \|\tilde{x}_k\|$. The fact that the norm is positive semidefinite and $\gamma \in [0, 1)$ necessitates that $\limsup_{k\to\infty} \|\tilde{x}_k\| = 0$. Thus, x converges to $\bar{x}$. Similarly, by (11) we have

$$\begin{aligned} \limsup_{k\to\infty} \|\tilde{z}_k\| \le &\left(\|\bar{P}\| + \rho\right) \limsup_{k\to\infty} \|\tilde{x}_k\| \\ &+ \limsup_{k\to\infty} \left(\|\tilde{P}_k\|\|\bar{x}\| + \|\tilde{r}_k\| + \|d_k\|\right). \end{aligned}$$

The convergence assumptions on the uncertain terms imply that the right hand side of the inequality tends to 0. Thus, z converges to $\bar{z}$. Therefore, by propositions 3 and 4, u and e converge to $\bar{u}$ and $\bar{e}$, respectively.

■

Remark 2: The existence of the limit superiors in the above analyses are guaranteed by the boundedness of the sequences by virtue of the monotone convergence theorem. In theorem 3, convergence of the uncertain terms imply their boundedness, hence $\tilde{x}$ is bounded.

Remark 3: Note that as opposed to [10], the convergence conditions are more relaxed since we do not require P, r, d to converge exponentially.

It is of course possible to carry out the same analysis without the auxiliary signals x and z. By using x and z, we would like to emphasize the fact that much like in linear time invariant systems or iteration invariant ILC systems, the initial condition has no effect in the limit.

VI. DESIGN CHOICES

In this section we briefly discuss how the design choices of L and γ affect system performance. In particular, we see that faster convergence is desired to minimize the variance of the system, along with lower gains on L.

A. Convergence Speed

As in the iteration invariant case, it is trivial to see that γ is a measure of the convergence speed of the algorithm: Recall from section IV that the input and error converge to the auxiliary variables that describe the "steady state" response (or forced response) of the ILC system. Furthermore, we saw in section III that the effect of the initial input vanishes exponentially with rate $-\ln\gamma$. Similarly, the various bounds throughout the paper can be shown to be attained with the same rate. Hence, lower values of γ correspond to a fast convergence to the forced response of the system, and vice versa.

B. Limiting Performance

We turn our attention to the asymptotic performance of the system. Noting again that $\bar{x}$ and $\bar{z}$ represent the forced response of the nominal system, by the proof of proposition 2 the bound $\|\bar{x}_\infty\| \leq \|L\|\|\bar{r}\|/(1-\gamma)$ holds. Thus, $\|\bar{x}_\infty\|$ increases as $\|L\|$ and/or γ increases. Further, we see the same relationship in (10) and (12). Therefore, the system converges to the nominal case as the gain of L and/or the convergence factor γ is decreased.

VII. ASIDE ON THE Q FILTER

In the previous section, we analyzed how the design choices of L and γ affect system performance. It seems that in general, a controller with low gains on L and a fast convergence (i.e. a low γ) is desired to obtain good asymptotic performance. So to achieve the lowest deviation from the nominal case, one should minimize $\|L\|/(1-\gamma)$. However, the relationship between L and γ is still unclear. Similarly, how the choice of Q affects the convergence factor γ is yet to be explained.

A typical algorithm in the literature is the Q filter and learning function approach, where

$$u_{k+1} = Q(u_k + \bar{L}e_k), \quad \forall k \in \mathbb{N}. \tag{13}$$

This algorithm can be seen as a special case of the linear algorithm (2) with $L = Q\bar{L}$[1]. This formulation has some advantages over the more general algorithm presented in the manuscript. For instance, the robustness condition of assumption 3 can be satisfied in any normed space by decreasing the gain of Q via the submultiplicativity of the induced norm. The algorithm is popular especially in frequency domain designs where well established heuristics describe the performance versus robustness trade-off imposed by Q [12], [13]. In other words, this approach enables an easy way to control the convergence factor γ. We discuss how zero error in the limit can be guaranteed, and how the Q filter affects performance when L can be factored as $Q\bar{L}$.

[1]It is shown in [11] that for the finite dimensional case the two algorithms are equivalent when designed by the norm optimal method.

A. Driving the Error to the Origin

It is well known that for a monotonically convergent linear algorithm where $\mathcal{U} = \mathcal{Y}$ is the Cartesian product of $\mathbb{R}$ or $\mathcal{H}_2$, the converged error equals 0 for every reference if and only if $Q = I$. We discuss a generalization of this theorem below.

Theorem 4: Given $\bar{r} \in \mathcal{Y}$, let $\bar{e}_\infty$ be the converged error of the nominal system. Define $\mathcal{U}_1$ as the eigenspace of Q corresponding to an eigenvalue of 1. Then, $\bar{e}_\infty = 0$ if and only if there exists $\bar{u}^* \in \mathcal{U}_1$ in the preimage of $\bar{r}$. Moreover, if such a $\bar{u}^*$ exists, it is unique and equal to the converged input $\bar{u}_\infty$.

Proof: Let $\bar{u}_\infty$ be the converged input of the system. Direct computation shows that

$$(I - Q)\bar{u}_\infty = L(\bar{r} - \bar{P}\bar{u}_\infty) = L\bar{e}_\infty,$$

hence $\bar{e}_\infty = 0$ if and only if $\bar{u}_\infty$ lies in $\mathcal{U}_1 \cap \bar{P}^{-1}\bar{r}$. Furthermore, the contraction mapping theorem states that $\bar{u}_\infty$ must be the unique fixed point of the update law, eliminating the possibility that $\mathcal{U}_1 \cap \bar{P}^{-1}r$ has more than a single element. ∎

Consequently, $\bar{P}$ and $L\bar{P}$ are nonsingular or equivalently injective in $\mathcal{U}_1$: If this was false, there would be an element $\bar{u}$ such that $\|(Q - L\bar{P})\bar{u}\| = \|\bar{u}\|$, contradicting $\|Q - L\bar{P}\| < 1$. Note that for a bijective map the condition to ensure zero error for all references simplifies to $Q = I$ as expected. This generalization is interesting as it relates to operators that are not necessarily bijective. A necessary and sufficient condition for converging to arbitrary references in $\mathcal{Y}$ via the contraction mapping based update is to find a subspace $\mathcal{U}_1 \subseteq \mathcal{U}$ such that $Q|_{\mathcal{U}_1}$ is the identity and $\bar{P}(\mathcal{U}_1) = \mathcal{Y}$; which necessitates that $\bar{P}|_{\mathcal{U}_1}$ and $L\bar{P}|_{\mathcal{U}_1}$ are bijective. For instance, for a redundant system, this means that Q cannot be the identity map. An example follows below.

Example 1: Consider $\mathbb{R}^2$ and $\mathbb{R}$ equipped with the sup norm. Let $\bar{P} = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Take

$$Q = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix},$$

which yields $\|\bar{T}\|_\infty = 1/2$. From (7), it is straightforward to check that for any $\bar{r} \neq 0$, $\bar{u}_\infty = \bar{r}\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ attains $\bar{e}_\infty = 0$ and is the eigenvector of Q corresponding to the eigenvalue 1. It is also easy to verify that when $Q = I$, the spectral radius of $\bar{T}$ is bounded below by 1; i.e. the contraction condition can never be satisfied for any norm since $\bar{P}$ is not injective.

For a nonsurjective map, the best we can hope is to achieve zero error for references in the image of $\bar{P}$. If the map is injective, Q must necessarily be the identity and we can conveniently select L as a left inverse of $\bar{P}$.

Example 2: Consider $\mathbb{R}$ and $\mathbb{R}^2$ equipped with the 2 norm. Let $\bar{P} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Take $Q = 1$ and $L = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$, the Moore-Penrose pseudoinverse of $\bar{P}$, which leads to $\bar{T} = 0$. From (7), $\bar{u}_\infty = L\bar{r}$ attains $\arg\min_{\bar{u}\in\mathbb{R}} \|\bar{r} - \bar{P}\bar{u}\|_2$ in a single trial which obviously equals 0 if and only if $\bar{r}$ is in the range space of $\bar{P}$.

Henceforth, we will refer to monotonically convergent algorithms that guarantee 0 error for references in the image of $\bar{P}$ as *iterative integrators*.

B. Using the Q Filter to Achieve Robustness

We consider the case when L can be factored as $Q\bar{L}$. A necessary and sufficient condition for the existence of such a factorization is that $L(\mathcal{Y}) \subseteq Q(\mathcal{U})$. Two cases of interest are discussed:

1) Iterative Integrators: The condition for the $Q\bar{L}$ factorization is trivially satisfied when $\bar{P}$ is injective as this requires $Q = I$. In this case, algorithms (2) and (13) are equivalent with $\bar{L} = L$. If the image of L is contained in $\mathcal{U}_1$[2], $\mathcal{U}_1$ can be shown to be an invariant subspace of the nominal system. With this assumption, generally speaking, we can assume $\bar{L} = L$ for an iterative integrator. Therefore, for these algorithms, the worst case deviation from the nominal system in the limit can be described by substituting $\|\bar{x}_\infty\| = 0$ to (10) and (12), with no observable guidelines on how to optimize performance: The minimization of $\|L\|/(1-\gamma)$ is the key to low variance and will be specific to each problem.

2) Nonzero Asymptotic Errors: We discuss the more general case where $\mathcal{U}_1 = \{0\}$, assuming without loss of generality that the following condition holds:

$$\gamma = \sup_{H\in\mathcal{P}} \|Q(I - \bar{L}H)\| \leq \|Q\| \sup_{H\in\mathcal{P}} \|I - \bar{L}H\| < 1.$$

The robustifying effect of the Q filter in algorithm (13) can be seen in the above equation: By decreasing the gain of Q, γ can be rendered arbitrarily small to satisfy the contraction condition. Moreover, if we let $\bar{\gamma} \triangleq \sup_{H\in\mathcal{P}} \|I - \bar{L}H\|$, then the following inequality is valid since $L = Q\bar{L}$:

$$\frac{\|L\|}{1-\gamma} \leq \frac{\|Q\|\|\bar{L}\|}{1-\|Q\|\bar{\gamma}}.$$

Hence the performance measure $\|L\|/(1-\gamma)$ can be decreased by decreasing $\|Q\|$. It turns out that variance of the iteration varying system can be controlled by the gain of Q, with decreasing gains signifying decreasing variance.

VIII. CONCLUSIONS

In this paper, we scrutinized the stability and convergence properties of ILC systems subject to trial to trial uncertainty. We formulated the system to be controlled as a linear input-output map in an abstract Banach space setting to ensure the generality of our analysis, assuming bounded uncertainties in all process parameters; including the input-output operator, the feedback response, reference, noise, disturbance and initial conditions. We showed that when a linear update law is designed to be robust over the set of possible maps $\mathcal{P}$, LDT methods can be employed directly to show the system exhibits desirable properties such as asymptotic stability and boundedness. Furthermore, we discussed how the design of the operators Q and L affects the convergence properties of iteration varying systems. We showed that an iteration varying system converges to 1) a bounded neighborhood of a nominal system if the uncertainties are bounded, and, 2) the nominal system itself if the uncertainties are convergent.

It turns out that robust ILC methods, which are well studied in the literature [13], [14], [15], [16] (see also the references in [17]), can be applied directly to iteration varying systems. The results are quite strong in terms of their generality and the lack of limiting assumptions apart from boundedness and linearity. We expect these initial results to pave the way for increased application diversity of ILC with relaxed assumptions, along with the supporting theory.

[2]An easy way of ensuring this is composing L with a projection operator with kernel $\mathcal{U}_0$ so that $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_0$. If Q is also chosen as this projection, then the algorithm would converge to $\mathcal{U}_1$ in a single step.

REFERENCES

[1] S. Arimoto, S. Kawamura, and F. Miyazaki, "Bettering operation of robots by learning," *Journal of Robotic Systems*, vol. 1, no. 2, pp. 123–140, 1984.

[2] D. Bristow, M. Tharayil, and A. Alleyne, "A survey of iterative learning control," *Control Systems, IEEE*, vol. 26, no. 3, pp. 96–114, 2006.

[3] K. L. Moore, *Iterative Learning Control for Deterministic Systems*. London: Springer-Verlag, 1993.

[4] Z. Hou, J. Yan, J.-X. Xu, and Z. Li, "Modified iterative-learning-control-based ramp metering strategies for freeway traffic control with iteration-dependent factors," *Intelligent Transportation Systems, IEEE Transactions on*, vol. 13, no. 2, pp. 606–618, June 2012.

[5] M. Norrlöf and S. Gunnarsson, "Time and frequency domain convergence properties in iterative learning control," *International Journal of Control*, vol. 75, no. 14, pp. 1114–1126, 2002.

[6] J.-X. Xu and J. Xu, "On iterative learning from different tracking tasks in the presence of time-varying uncertainties," *Systems, Man, and Cybernetics, Part B: Cybernetics, IEEE Transactions on*, vol. 34, no. 1, pp. 589–597, Feb 2004.

[7] D. Bristow, "Optimal iteration-varying iterative learning control for systems with stochastic disturbances," in *American Control Conference (ACC), 2010*, June 2010, pp. 1296–1301.

[8] C. Yin, J.-X. Xu, and Z. Hou, "A high-order internal model based iterative learning control scheme for nonlinear systems with time-iteration-varying parameters," *Automatic Control, IEEE Transactions on*, vol. 55, no. 11, pp. 2665–2670, Nov 2010.

[9] Y.-Q. Chen and K. Moore, "Harnessing the nonrepetitiveness in iterative learning control," in *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, vol. 3, Dec 2002, pp. 3350–3355 vol.3.

[10] D. Meng and K. Moore, "On robust iterative learning control against iteration-varying uncertain plant parameters," in *American Control Conference (ACC), 2014*, June 2014, pp. 4255–4261.

[11] D. Bristow and B. Hencey, "A Q, L factorization of norm-optimal iterative learning control," in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, Dec 2008, pp. 2380–2384.

[12] D. A. Bristow, K. L. Barton, and A. G. Alleyne, "Iterative learning control," in *The Control Handbook*, W. S. Levine, Ed. Salem, MA: CRC Press, 2010.

[13] D. De Roover and O. H. Bosgra, "Synthesis of robust multivariable iterative learning controllers with application to a wafer stage motion system," *International Journal of Control*, vol. 73, no. 10, pp. 968–979, 2000.

[14] J. van de Wijdeven, T. Donkers, and O. Bosgra, "Iterative learning control for uncertain systems: Robust monotonic convergence analysis," *Automatica*, vol. 45, no. 10, pp. 2383 – 2391, 2009.

[15] H.-S. Ahn, K. L. Moore, and Y. Chen, "Stability analysis of discrete-time iterative learning control systems with interval uncertainty," *Automatica*, vol. 43, no. 5, pp. 892 – 902, 2007.

[16] B. Altın and K. Barton, "Robust iterative learning for high precision motion control through $\mathcal{L}_1$ adaptive feedback," *Mechatronics*, vol. 24, no. 6, pp. 549 – 561, 2014, control of High-Precision Motion Systems.

[17] H.-S. Ahn, Y.-Q. Chen, and K. Moore, "Iterative learning control: Brief survey and categorization," *Systems, Man, and Cybernetics, Part C: Applications and Reviews, IEEE Transactions on*, vol. 37, no. 6, pp. 1099–1121, 2007.