On Linearized Stability of Differential Repetitive Processes and Iterative Learning Control*

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Abstract— **Repetitive processes are two dimensional (2D) systems that arise in the modeling of engineering applications such as additive manufacturing, in which information propagation occurs along two axes of independent variables. While the existing literature on repetitive processes is predominantly on linear systems, recent work highlights the need to develop rigorous tests for stability of nonlinear processes. Using existing results from linear repetitive process theory, we establish a differential repetitive process analogue of the well known result that the stability of a nonlinear feedback system can be verified by the stability of the linearized dynamics. In particular, we employ a 2D Lyapunov equation to show that the feasibility of a linear matrix inequality, combined with 2 small gain conditions, can guarantee stability locally around an equilibrium. Finally, we apply this result to the design and stability analysis of iterative learning control (ILC) systems, and discuss implications in the context of nonlinear ILC.**

I. INTRODUCTION

Repetitive processes are two dimensional (2D) dynamic systems that arise in the modeling of engineering applications such as additive manufacturing, in which information propagation occurs along two axes of independent variables. These processes are characterized by a series of sweeps, termed passes (coincidentally, the processes were also named multipass earlier in the literature), with finite length or duration, that act as forcing functions on the dynamics of future passes [1]. A closely related field is iterative learning control (ILC), which can be thought of as a special class of repetitive processes, wherein the pass to pass dynamics are induced through the construction of a recurrence relation that updates the feedforward input using past data.

The literature on repetitive processes (and other 2D systems) is predominantly on linear systems. While much work has been done in this area in terms of stability analysis and control synthesis, recent work highlights the need to develop rigorous tests for stability of nonlinear processes [2], [3], [4]. In the ILC literature, it has been noted that nonlinear update laws have not been extensively researched, save for adaptive laws for locally Lipschitz plants, and a systematic theory of nonlinear ILC is an open question [5], [6], [7]. The need to develop a sound theory for nonlinear repetitive processes is further supported by practical applications of a repetitive nature such as laser metal deposition [8], [9].

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In this paper, using existing results from linear repetitive process theory, we establish a differential repetitive process analogue of the well known result that the stability of a nonlinear one dimensional (1D) feedback system can be easily verified by the stability of the linearized dynamics. In particular, we employ a 2D Lyapunov equation to show that the feasibility of a linear matrix inequality (LMI), combined with 2 small gain conditions on the pass to pass transfer function, can guarantee stability locally around an equilibrium. Finally, we apply this result to the design and stability analysis of ILC systems, and discuss implications in the context of nonlinear ILC.

The rest of the paper is organized as follows: Section II gives some theoretical background on linear repetitive processes. In section III, we introduce state space representations of linear time invariant (LTI) differential repetitive processes, along with the 2D Lyapunov equation. A representation of nonlinear time invariant repetitive processes is presented in section IV, with conditions on stability, followed by our main result. Section V briefly discusses the implications of our result in the context of nonlinear ILC. An illustrative example is given in section VI, with concluding remarks in section VII. For a more streamlined presentation, two technical results that are used in the proof of our main theorem are given in the appendix.

II. BACKGROUND AND PRELIMINARIES

In this section, we give a brief background on the general model of linear repetitive processes, and present the style of notation to be used throughout the manuscript for clarity.

A. Notation

We use $\mathbb{R}$ to represent the set of real numbers, and $\mathbb{N}$ the set of nonnegative integers. The spectral radius of a linear operator H is denoted $\rho(H)$. For a matrix H, H^T is the transpose, and $H > 0$ implies positive definiteness. Given square matrices H_1 and H_2, $H_1 \oplus H_2$ is their direct sum. I and 0 denote the identity and zero matrices of appropriate size, respectively. Given a transfer function $H(s)$, $\|H(s)\|_\infty$ is the $\mathcal{H}_\infty$ norm. For a real vector h, $\|h\|_\infty$ and $\|h\|_2$ are the sup and the 2 norms, respectively. $\mathcal{L}_p$ denotes the space of piecewise continuous functions $\mathbb{R} \mapsto \mathbb{R}^l$ of any dimension l with finite $\mathcal{L}_p$ norm, $p \in [1, \infty]$; $\mathcal{L}_p[0,T]$ will be used to denote the subspace such that functions have support $[0,T]$. Specifically, for $h : \mathbb{R} \to \mathbb{R}^l$, $\|h\|_{\mathcal{L}_\infty} \triangleq \sup_{t\in\mathbb{R}} \|h(t)\|_\infty$, and similarly, $\|h\|_{\mathcal{L}_2} \triangleq \sqrt{\int_{-\infty}^{\infty} \|h(\tau)\|_2^2 \, d\tau}$. For Banach spaces X_1 and X_2, $\mathcal{B}(X_1, X_2)$ is the space of all bounded linear operators mapping X_1 to X_2. $\jmath$ is the imaginary unit. $o(.)$ denotes the asymptotic little o notation.

B. Linear Repetitive Processes in Banach Space

A general abstract model of a linear repetitive process assumes an underlying Banach space structure [1]. In particular, we assume that the output at pass (or iteration) k, denoted y_k, is a vector in a subspace Y_T of a complete function space Y, where $T < \infty$ denotes the duration or length of the pass profile. Then,

$$y_{k+1} = L_T y_k + u_{k+1}, \quad \forall k \in \mathbb{N} \tag{1}$$

where $L_T \triangleq L|_{Y_T}$ is the restriction[1] of $L \in \mathcal{B}(Y, Y)$, and u_k is a vector in a subspace $U_T \subseteq Y_T$ that represents the effect of initial conditions, disturbance, noise, and the control input. A concrete example of the abstract formulation (1) will be given in the next section through linear differential processes.

Definition 1: The linear repetitive process (1) is *asymptotically stable* if there exists a positive number υ such that for any $y_0 \in Y_T$ and any strongly convergent sequence $\{u_k\}_{k=1}^{\infty}$, the output $\{y_k\}_{k=1}^{\infty}$ generated by the perturbed system

$$y_{k+1} = (L_T + \Delta) y_k + u_{k+1}, \quad \forall k \in \mathbb{N}$$

is strongly convergent for all $\Delta \in \mathcal{B}(Y_T, Y_T)$ with $\|\Delta\| < \upsilon$.

Theorem 1 ([1], pg. 44): The linear repetitive process (1) is asymptotically stable if and only if $\rho(L_T) < 1$.

As a result of the above theorem, there exist scalars $M_T > 0$ and $\gamma_T \in [0, 1)$ such that for any constant sequence $u_k = u$ for all $k \in \mathbb{N}$

$$\|y_k - y_\infty\| \leq M_T \gamma_T^k \left(\|y_0\| + \frac{\|u\|}{1 - \gamma_T} \right), \quad \forall k \in \mathbb{N},$$

where y_∞ is the strong limit of the sequence $\{y_k\}_{k=1}^{\infty}$.

A stronger notion of stability for these processes is that of *stability along the pass*, in which we require that there exist scalars M and $\gamma \in [0, 1)$ such that $M_T \leq M$ and $\gamma_T \leq \gamma$ for all $T \in [0, \infty)$. The conditions for stability along the pass of (1) is omitted here; we merely note that this notion translates to differential repetitive processes as requiring the dynamics governing the evolution of the state to be exponentially stable, as we will see in the next section.

III. STATE SPACE FORMULATION OF LINEAR DIFFERENTIAL REPETITIVE PROCESSES

LTI differential repetitive processes without exogenous inputs can be represented in state space form as

$$\begin{aligned} \dot{x}_{k+1}(t) &= A x_{k+1}(t) + B y_k(t), \\ y_{k+1}(t) &= C x_{k+1}(t) + D y_k(t), \end{aligned} \tag{2}$$

for all $t \in [0, T]$, $k \in \mathbb{N}$, where $x_k(t) \in \mathbb{R}^n, y_k(t) \in \mathbb{R}^m$ are the state and output vectors, respectively, and A, B, C, D are real matrices of appropriate size which form a minimal realization. Note that it is also necessary to specify boundary conditions $y_0 \in \mathcal{L}_\infty[0, T]$ and bounded $\{x_{k+1}(0)\}_{k=0}^{\infty}$ to uniquely determine the solution.

[1]Here, we also restrict the codomain of L to Y_T via some truncation like operation.

The state space formulation (2) represents only a subset of the recursion relations that can be defined on the underlying function space $(\mathcal{L}_2 \cap \mathcal{L}_\infty)[0, T]$, specifically the class of systems in which the output at iteration k acts in a point-wise manner on the state derivative and output vectors at pass $k+1$. As a counter example, consider an ILC algorithm in which the input is synthesized in a noncausal manner. Clearly, (2) is a highly inaccurate representation for such a system since information from y_k is transmitted to y_{k+1} only through a causal integration. Similar representations can be developed for a variety of systems; e.g. discrete processes, iterative algorithms for nonlinear optimal control problems [1].

A. Stability

Before stating the basic stability conditions and the 2D Lyapunov function that will be used in nonlinear stability analysis, we define the augmented state matrix Φ and the pass to pass transfer function $G(s)$ of the quadruple (A, B, C, D):

$$\Phi \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad G(s) \triangleq C(sI - A)^{-1} B + D.$$

We say that Φ is *stable along the pass* if its corresponding linear state space representation is stable along the pass. Similarly, we say that Φ is *Hurwitz* if the corresponding matrix A is Hurwitz. Stability conditions of the linear system are given below.

Theorem 2 ([1], pg. 49): The linear repetitive process (2) is asymptotically stable if and only if D is Schur.

Theorem 3 ([1], pg. 62): The linear repetitive process (2) is stable along the pass if and only if A is Hurwitz, D is Schur, and $\rho(G(\jmath\omega)) < 1$ for all $\omega \in \mathbb{R}$.

B. The 2D Lyapunov Equation

The 2D Lyapunov equation for linear differential repetitive processes is given by the following formula:

$$\Phi^T W^{1,0} + W^{1,0} \Phi + \Phi^T W^{0,1} \Phi - W^{0,1} = -Q = -Q^T, \tag{3}$$

where $W^{1,0} \triangleq W_1 \oplus 0$ and $W^{0,1} \triangleq 0 \oplus W_2$, and the matrices $W_1 = W_1^T \in \mathbb{R}^{n \times n}$ and $W_2 = W_2^T \in \mathbb{R}^{m \times m}$. At first glance, it seems that (3) is a combination of 1D continuous and discrete time algebraic Lyapunov equations. However, because of the 2D structure of the system, the two different LMIs corresponding to these cases cannot be used to solve (3) (see pg. 169 of [1]). The use of (3) in stability analysis is given by the following theorem.

Theorem 4 ([1], pg. 130): The linear repetitive process (2) is stable along the pass if there exist positive definite matrices W_1 and W_2 such that the solution Q of the 2D Lyapunov equation (3) is positive definite.

In contrast to the 1D case, the conditions of theorem 4 are only sufficient for stability, and the eigenvalue conditions on the system matrices A and D do not guarantee the existence of $W_1 > 0$ and $W_2 > 0$ that yield $Q > 0$, save for single input single output (SISO) systems. However, we can circumvent this issue by requiring the slightly stronger condition that $\|G(s)\|_\infty < 1$, as is typical in contraction mapping based ILC with monotonically convergent algorithms.

Lemma 1: For a stable along the pass linear repetitive process of the form (2), if $\|G(s)\|_\infty < 1$, then there exist $W_1, W_2 > 0$ such that the solution Q of the 2D Lyapunov equation (3) is positive definite.

Proof: This follows directly from theorem 4.2.1 of [1]. ■

IV. STABILITY OF NONLINEAR DIFFERENTIAL REPETITIVE PROCESSES

In this section, we will present the main result of our manuscript: First, we introduce the mathematical representation of nonlinear time invariant differential repetitive processes, along with the definition of exponential stability for these systems. We then restate the Lyapunov theorem from [2] and proceed with the linearization of the nonlinear system around the origin. Next, we introduce an $\mathcal{L}_1$ norm condition which constrains the trajectories of the system to a neighborhood of Euclidean space. Finally, using the asymptotic property of the linear approximation, along with the $\mathcal{L}_1$ norm condition, we show that we can ensure that the Lyapunov function for the linear system is a local Lyapunov function for the nonlinear system, thereby concluding stability.

Consider the following nonlinear model of a time invariant differential repetitive process without exogenous inputs, with the same state and ouput dimensions as the linear case:

$$\begin{aligned} \dot{x}_{k+1}(t) &= f(x_{k+1}(t), y_k(t)), \\ y_{k+1}(t) &= g(x_{k+1}(t), y_k(t)), \end{aligned} \tag{4}$$

for all $t \in [0, T]$ and $k \in \mathbb{N}$ in $\mathbb{R}^n \times \mathbb{R}^m$. Here, f and g are continuously differentiable functions that vanish at the origin. We also assume $\|x_{k+1}(0)\|_\infty \le \kappa_x \zeta^k$ for all $k \in \mathbb{N}$ and $\|y_0\|_{\mathcal{L}_\infty} \le \kappa_y$, for some $\kappa_x, \kappa_y > 0$, and $\zeta \in (0, 1)$.

Definition 2: The system is said to be *exponentially stable* (in the $\mathcal{L}_2$ sense) if there exist scalars K_x, K_y such that $\kappa_x < K_x$ and $\kappa_y < K_y$ imply $\|y_k\|_{\mathcal{L}_2} \le K(\zeta)\chi(\zeta)^k$ for all $k \in \mathbb{N}$, for some $K : [0, 1) \to \mathbb{R}$ and $\chi : [0, 1) \to [0, 1)$.

Remark 1: As opposed to [2], the conditions on the boundaries are set in terms of the sup norm instead of the 2 norm even though exponential stability is defined in the $\mathcal{L}_2$ norm topology. This is rather for convenience and fortunately does not contradict the 2 norm based conditions of [2] since all norms in finite dimensions are equivalent.

A. Exponential Stability via Lyapunov Functions

The stability of the equilibrium can be assessed via Lyapunov functions as shown in [2]: We let $V_1 : \mathbb{R}^n \to \mathbb{R}$, and $V_2 : \mathbb{R}^m \to \mathbb{R}$ be positive definite functions such that $V_1(0) = V_2(0) = 0$. We define the candidate Lyapunov function for the system as $V(x, y) \triangleq (V_1(x), V_2(y))$. The *divergence* of this function along the trajectories of (4) is defined as

$$\begin{aligned} \operatorname{div}\left(V(x_{k+1}(t), y_k(t))\right) \triangleq\ & \dot{V}_1(x_{k+1}(t)) \\ & + \left(V_2(y_{k+1}(t)) - V_2(y_k(t))\right), \end{aligned}$$

for all $k \in \mathbb{N}$ and $t \in [0, T]$.

Theorem 5 ([2]): The nonlinear differential repetitive process (4) is exponentially stable in $\mathcal{L}_2$ if there exists a Lyapunov function V with constants c_1, c_2, c_3, with $c_2 > c_3$, such that

$$\begin{aligned} c_1 \|x\|_2^2 \le V_1(x) \le c_2 \|x\|_2^2, \\ c_1 \|y\|_2^2 \le V_2(y) \le c_2 \|y\|_2^2, \end{aligned} \tag{5}$$

and

$$\operatorname{div}(V(x, y)) \le -c_3 \|(x, y)\|_2^2. \tag{6}$$

along the trajectories of (4).

B. Linearized Dynamics

We proceed with the linearization of (4) as follows. Since f and g are continuously differentiable, application of the multivariable mean value theorem to (4) means

$$\begin{aligned} \dot{x}_{k+1}(t) &= \bar{A}x_{k+1}(t) + \bar{B}y_k(t) + b(x_{k+1}(t), y_k(t)), \\ y_{k+1}(t) &= \bar{C}x_{k+1}(t) + \bar{D}y_k(t) + d(x_{k+1}(t), y_k(t)), \end{aligned} \tag{7}$$

for all $t \in [0, T]$ and $k \in \mathbb{N}$, where

$$\begin{aligned} \bar{A} &= \frac{\partial f}{\partial x}(0), \quad & \bar{B} &= \frac{\partial f}{\partial y}(0), \\ \bar{C} &= \frac{\partial g}{\partial x}(0), \quad & \bar{D} &= \frac{\partial g}{\partial y}(0), \end{aligned}$$

and $b(x, y), d(x, y)$ are continuous functions such that both $\|b(x, y)\|, \|d(x, y)\|$ are $o(\|(x, y)\|)$ as $(x, y) \to 0$ in any norm $\|.\|$. We define $\bar{\Phi}$ and $\bar{G}(s)$ to be the augmented state matrix and the pass to pass transfer function, respectively, of the quadruple $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$. The matrix $\bar{\Phi}$ will also be called the *linearization* of (4). In addition, we define the function $\varphi(x, y) \triangleq (b(x, y), d(x, y))$.

C. Local Stability Analysis through Linearized Dynamics

In analyzing the stability of the nonlinear system through its linearization, we will need an additional condition that will aid us by constraining the state and input vectors to neighborhoods of the origin in Euclidean space. This condition is stated in terms of the $\mathcal{L}_1$ norm of a transfer function (essentially its *induced* $\mathcal{L}_\infty$ *norm*; see lemmas A.7.1 and A.7.2 of [10]), whose definition is given below. The necessary result concerning the use of the $\mathcal{L}_1$ norm is given in the appendix (lemma 3), and can be interpreted as a contraction or a small gain condition in the pass domain.

Definition 3: The $\mathcal{L}_1$ *norm* of a q input r output LTI system $H(s)$ with impulse response $h(t) \in \mathbb{R}^{q \times r}$ is defined as $\|H(s)\|_{\mathcal{L}_1} \triangleq \max_{i \in 1,2,\dots,r} \sum_{j=1}^{q} \|h_{ij}\|_{\mathcal{L}_1}$, where $h_{ij}(t)$ is the entry at the ith row and jth column of $h(t)$.

We are now ready to state our main result.

Theorem 6: The nonlinear differential repetitive process (4) is exponentially stable if its linearization $\bar{\Phi}$ is stable along the pass, minimal, and the associated pass to pass transfer function $\bar{G}(s)$ satisfies $\|\bar{G}(s)\|_\infty < 1$ and $\|\bar{G}(s)\|_{\mathcal{L}_1} < 1$.

Proof: The proof relies on the construction of an appropriate Lyapunov function. Specifically, since the linearization is stable along the pass and minimal, and the pass to pass transfer function satisfies the $\mathcal{H}_\infty$ contraction condition, we know from lemma 1 that there exist positive definite matrices W_1 and W_2 such that the solution Q of the 2D Lyapunov equation (3) is positive definite. Hence, let $V_1(x) = x^T W_1 x$ and $V_2(y) = y^T W_2 y$. Then, V_1 and V_2 satisfy (5) globally with c_1 as the minimum of their eigenvalues, and c_2 as the maximum of their eigenvalues. Also note that c_2 can be increased if necessary to satisfy the condition $c_2 > c_3$. Now define $z \triangleq (x, y)$. Readers can then check that the divergence of V satisfies

$$\operatorname{div}(V(z)) = -z^T Q z$$
$$+2\left(\varphi^T(z)\underbrace{\begin{bmatrix} W_1 & 0 \\ W_2 C & W_2 D\end{bmatrix}}_{\Lambda} z + \varphi^T(z)\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & W_2\end{bmatrix}}_{\Psi}\varphi(z)\right).$$

The above equation is indefinite in $\mathbb{R}^n \times \mathbb{R}^m$. However, we can use the asymptotic property of the function φ to select an open subset containing 0 in which the divergence satisfies (6): For all $\epsilon > 0$, there exists $\delta > 0$ such that $\|z\|_2 < \delta$ implies $\|\varphi(z)\|_2 \leq \epsilon \|z\|_2$. Then from above, we have

$$\operatorname{div}(V(z)) \leq \|z\|_2^2 \left(2\epsilon^2 \bar{\sigma}(\Psi) + 2\epsilon\bar{\sigma}(\Lambda) - \underline{\sigma}(Q)\right),$$

for all z with $\|z\|_2 < \delta$, where Ψ and Λ are matrices defined above, and $\bar{\sigma}(.)$ and $\underline{\sigma}(.)$ denote the maximum and minimum singular values, respectively. Therefore, if we take ϵ to be small enough, we can find a δ so that $\operatorname{div}(V(z)) \leq -c_3 \|z\|_2^2$ for $z \in \mathbb{R}^m \times \mathbb{R}^n$ with $\|z\|_2 < \delta$.

It remains to find positive constants K_x, K_y such that if $\|x_{k+1}(0)\|_\infty < K_x \zeta^k$ and $\|y_0\|_{\mathcal{L}_\infty} < K_y$, the trajectories of (4) will not leave the open ball of radius δ centered at $0 \in \mathbb{R}^n \times \mathbb{R}^m$. Since $\|\bar{G}(s)\|_{\mathcal{L}_1} < 1$, by lemma 3, if we choose $\bar{\delta} = \delta/\sqrt{n+m}$, as $\|z\|_2 \leq \sqrt{n+m}\,\|z\|_\infty$ for all $z \in \mathbb{R}^n \times \mathbb{R}^m$, we can find $\bar{\kappa}_x$ and $\bar{\delta}_y$ so the trajectories of (4) stay in the δ ball, provided $\|x_{k+1}(0)\|_\infty < \bar{\kappa_x} < \bar{\delta}$ for all $k \in \mathbb{N}$ and $\|y_0\|_{\mathcal{L}_\infty} < \bar{\kappa_x} < \bar{\delta}$. Thus, selecting $K_x = \bar{\kappa}_x$ and $K_y = \bar{\delta}_y$, by theorem 5, the system is exponentially stable.

■

Remark 2: For SISO systems, it suffices to check the condition $\|\bar{G}(s)\|_{\mathcal{L}_1} < 1$ since $\|\bar{G}(s)\|_\infty \leq \sqrt{m}\,\|\bar{G}(s)\|_{\mathcal{L}_1}$. See lemma 2 of [11].

Remark 3: The $\mathcal{L}_1$ norm condition can be reformulated so the absolute integral of the impulse is computed only on $[0, T]$, to yield less conservative results.

Lemma 3 is crucial in proving our result: Although we can find a region in $\mathbb{R}^n \times \mathbb{R}^m$ such that the Lyapunov function satisfies the necessary conditions, it is only by lemma 3 that we can guarantee that the trajectories stay uniformly in this region, for all $t \in [0, T]$ and $k \in \mathbb{N}$. The necessity of this additional condition boils down to the fact that $\mathcal{L}_p$ spaces are infinite dimensional, and consequently convergence in $\mathcal{L}_2$ does not imply convergence in $\mathcal{L}_\infty$[2]; e.g. consider a sequence $h_{k+1}(t) = k+1$ for $t \in [0, T/(k+1)^4]$ and $h_{k+1}(t) = 0$ otherwise, for all $k \in \mathbb{N}$. Thus, we cannot use theorem 5 directly to ensure that the trajectories remain uniformly bounded. As a result, the proof of theorem 6 requires significantly more effort compared to its 1D counterpart (for example, theorem 4.7 of [12]). See the appendix for the details.

[2]The converse is true since $\|.\|_{\mathcal{L}_2} \leq \sqrt{Tm}\,\|.\|_{\mathcal{L}_\infty}$ in our problem.

V. APPLICATION TO NONLINEAR ILC DESIGN AND ANALYSIS

In this section, we have a brief look at how the findings of the last section applies to nonlinear ILC design and analysis. Consider the following linearization of a time invariant system:

$$\dot{x}_{k+1}(t) = \bar{A}x_{k+1}(t) + \bar{B}u_{k+1},$$
$$y_{k+1}(t) = \bar{C}x_{k+1}(t) + \bar{D}u_{k+1}.$$

We assume without loss of generality that $\bar{A}$ is Hurwitz, since for the opposite case we can stabilize $\bar{A}$ through state feedback provided $(\bar{A}, \bar{B})$ is controllable. Our objective is to drive the output y_k to 0. We consider a proportional update law whose linearization is $u_{k+1}(t) = \bar{Q}u_k + \bar{L}y_k$. Then the system can be written in the following repetitive process form:

$$\dot{x}_{k+1}(t) = \bar{A}x_{k+1}(t) + \bar{B}\begin{bmatrix}\bar{L} & \bar{Q}\end{bmatrix}\begin{bmatrix} y_k(t) \\ u_k(t)\end{bmatrix},$$
$$\begin{bmatrix} y_{k+1}(t) \\ u_{k+1}(t)\end{bmatrix} = \begin{bmatrix}\bar{C} \\ 0\end{bmatrix} x_{k+1}(t) + \begin{bmatrix}\bar{D} \\ I\end{bmatrix}\begin{bmatrix}\bar{L} & \bar{Q}\end{bmatrix}\begin{bmatrix} y_k(t) \\ u_k(t)\end{bmatrix}.$$

The resulting pass to pass transfer function $\Omega(s)$ is given as

$$\Omega(s) \triangleq \begin{bmatrix}\bar{G}(s) \\ I\end{bmatrix}\begin{bmatrix}\bar{L} & \bar{Q}\end{bmatrix},$$

where $\bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$ as before. Since A is Hurwitz, by submultiplicativity, the conditions of theorem 6 can be satisfied as $\begin{bmatrix}\bar{Q} & \bar{L}\end{bmatrix} \to 0$. Therefore, we can find a locally exponentially stable linear or nonlinear proportional update law for any nonlinear system whose linearization is controllable.

A couple of comments are in order here:

1) We observe that when $\bar{Q} = I$, which is the necessary and sufficient condition for perfect tracking, the conditions of theorem 6 cannot be satisfied as this means $\|\Omega(s)\|_{\mathcal{L}_1} \geq 1 + \|\bar{L}\|_{\mathcal{L}_1}$. This is relatively surprising, since it is shown in [13], [5] that convergence can be achieved for a globally Lipschitz system if $\|I - \bar{L}\bar{D}\| < 1$ for any induced norm $\|.\|$, albeit in the time weighted norm topology. Based on this, it might be possible to reformulate the induced norm condition of lemma 3 in terms of the time weighted norm. Yet another issue that would need to be addressed is whether for $\bar{Q} = I$, we can find $\bar{L}$ so that
$$\rho\left(\begin{bmatrix}\bar{D} \\ I\end{bmatrix}\begin{bmatrix}\bar{L} & I\end{bmatrix}\right) < 1.$$
If D is singular, then the above problem is infeasible. This problem is addressed for linear systems by using

the superposition principle to define new vectors such that the input does not appear in the state space [2].

2) In essence, "stabilization" of an ILC system does not make much sense since the control objective in ILC is uniform transient tracking, wherein the limit profiles of either the state, input, or the output would be time varying. In this scenario, translating the coordinates to ensure that the origin is the equilibrium does not help with our analysis, since the resulting dynamics would also be time varying. Theorem 6 needs to be extended to time varying systems for this scenario, but this requires the development of 2D time varying Lyapunov equations, which is outside the scope of this exploratory work.

VI. ILLUSTRATIVE EXAMPLE

We consider the dynamic model of an actuated pendulum

$$\bar{m}\bar{l}\ddot{\theta}_k + \bar{b}\dot{\theta}_k + \bar{m}\bar{g}\sin(\theta_k) = u_k/\bar{l},$$

where $\bar{m}$ is the mass of the bob, $\bar{l}$ is the length of the rod, θ_k is the angle between the rod and the vertical axis, $\bar{b}$ is the rotational damping coefficient, $\bar{g}$ is the gravitational constant, and u_k is the input torque. Selecting the state as $x_k = (\theta_k, \dot{\theta}_k)$ and the output as $y_k = \theta_k$, we can linearize the equations about the origin to obtain the pass to pass transfer function, denoted $\bar{H}(s)$ here:

$$\bar{H}(s) = \frac{1}{\bar{m}\bar{l}^2}\frac{1}{(s^2 + \bar{b}/(\bar{m}\bar{l})s + \bar{g}/\bar{l})}.$$

Since $\bar{H}(s)$ is stable for all positive system parameters, we can satisfy the small gain and Schur conditions of theorem 6 by selecting learning updates with low enough gain around the origin. Our aim is to stabilize the pendulum for a period of 10 seconds, i.e. $T = 10$. We choose the system parameters as $\bar{g} = 9.81\,\text{m/s}^2$, $\bar{m} = 0.1\,\text{kg}$, $\bar{b} = 0.0006\,\text{Ns/m}$, $\bar{l} = 0.3\,\text{m}$, giving us $\left\|\bar{H}(s)\right\|_\infty \approx 971.5$. On the other hand, because the pendulum is locally a harmonic oscillator as $\bar{b} \to 0$, the $\mathcal{L}_1$ norm of the system is going to be extremely high. Thus, we rely on the finite approximation $\int_0^T |\bar{h}(t)|\,dt \approx 117.1$, where $\bar{h}(t)$ is the impulse response of $\bar{H}(s)$ (see remark 3). Then, taking the nonlinear update law as

$$u_{k+1}(t) = 10^{-4}[9u_k(t) - 1y_k(t)(1 + y_k^2(t))],$$

we can satisfy the conditions of theorem 6. Note that the gain of the update law increases as we move away from the equilibrium due to the nonglobally Lipschitz squared term, which is likely to destabilize the system.

We first consider a scenario wherein the initial input u_0 is Gaussian white noise that is scaled after its realization, so that $\|u_0\|_{\mathcal{L}_\infty} \approx 7.89$, and $y_0 = 0$. Similarly, the initial state $x_{k+1}(0)$ is randomly selected from a uniform distribution on $[0, 0.1(0.62)^k]$ for each k. The resulting output $\mathcal{L}_2$ norm sequence is normalized as $\|y_k\|_{\mathcal{L}_2} / \|y_1\|_{\mathcal{L}_2}$ for each k. As expected, fig. 1 shows that for boundary conditions high in amplitude the system is unstable. To verify that the system is locally stable, we also conduct a Monte Carlo type simulation of 100 realizations. For each realization, κ_x and ζ are randomly chosen from uniform distributions over $[0, 0.1]$ and $[0, 1]$, respectively. In the same manner, u_0 and y_0 are randomly chosen as Gaussian white noise, scaled after their realizations so that $\max\{\|u_0\|_{\mathcal{L}_\infty}, \|y_0\|_{\mathcal{L}_\infty}\} < 4.27$. For each realized pair (κ_x, ζ), x_{k+1} is drawn from a uniform distribution on $[0, \kappa_x\zeta^k]$, for all $k \in \mathbb{N}$ [3]. The resulting output sequences are normalized as before. Indeed, as theorem 6 predicts, we see in fig. 2 that the ILC system is locally exponentially stable, for the boundary parameters $K_x = 0.1$ and $K_y = 4.27$.

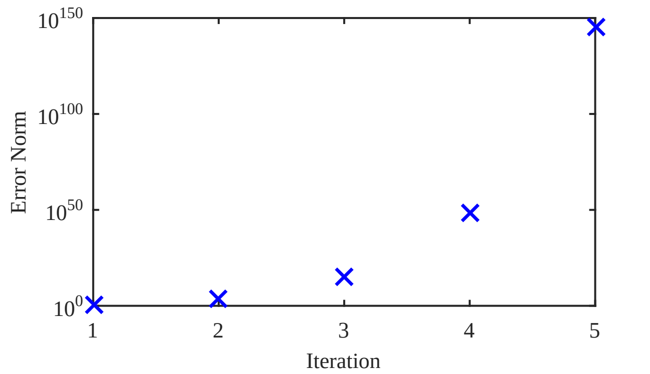

Fig. 1. Instability of nonlinear ILC with the parameters $\kappa_x = .1$, $\zeta = 0.62$, and $\|u_0\|_{\mathcal{L}_\infty} \approx 7.89$. The boundary conditions were randomly selected within the constraints imposed by the parameters.

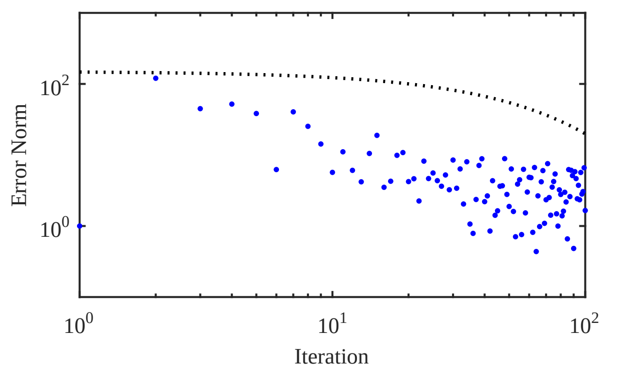

Fig. 2. Locally exponentially convergent behavior of nonlinear ILC over 100 realizations. The blue data points depict the maximum normalized errors at iteration k, respectively, over all realizations. The dashed black curve is the exponential decay function $150(0.98)^k$ for all $k \in \mathbb{N}$; in other words, the exponential convergence parameters $K(\zeta) < 150$ and $\chi(\zeta) < 0.98$ over all realized ζ.

VII. CONCLUSIONS

We established a differential repetitive processes analogue of the well known result that the stability of a nonlinear feedback system can be verified by the stability of the Jacobian matrix, using the 2D Lyapunov equation and certain small gain conditions. We also discussed the relevance of this result as it applies to nonlinear ILC. Future work will extend the results to different classes of systems.

[3]The objective of this is to validate that stability holds in a neighborhood of the origin, and not just for a specific initial state sequence and input.

APPENDIX

We present two lemmas that are helpful in evaluating local stability. The first establishes bounds on the state vector in terms of the boundary condition and the output, and will be used in the proof of the second.

Lemma 2: If the linearization $\bar{\Phi}$ of the nonlinear differential repetitive process (4) is Hurwitz, then there exist α, β positive such that $\|x_{k+1}\|_{\mathcal{L}_\infty} \leq \alpha \|y_k\|_{\mathcal{L}_\infty} + \beta \|x_{k+1}(0)\|_\infty$ in a neighborhood of the origin in $\mathbb{R} \times \mathcal{L}_\infty$.

Proof: This is a result of the small signal finite gain $\mathcal{L}_p$ stability theorem 5.1 of [12]. More specifically, the differential equation $\dot{x}_{k+1}(t) = f(x_{k+1}(t), y_k(t))$ satisfies all conditions of corollary 5.1 of [12] since 1) it is time invariant, and 2) the submatrix $\bar{A}$ of $\bar{\Phi}$ is Hurwitz. ■

The next lemma gives a uniform boundedness type result that is used in the proof of theorem 6.

Lemma 3: Assume that the linearization $\bar{\Phi}$ of (4) is Hurwitz, and its associated pass to pass transfer function satisfies $\mu \triangleq \|\bar{G}(s)\|_{\mathcal{L}_1} < 1$. Then, for all $\bar{\delta} > 0$, there exist positive constants $\bar{\delta}_y, \bar{\kappa}_x < \bar{\delta}$ such that if the boundary conditions satisfy $\|y_0\|_{\mathcal{L}_\infty} < \bar{\delta}_y$ and $\|x_{k+1}(0)\|_\infty < \bar{\kappa}_x$ for all $k \in \mathbb{N}$, we have $\|y_{k+1}\|_{\mathcal{L}_\infty} < \bar{\delta}$ and $\|x_{k+1}\|_{\mathcal{L}_\infty} < \bar{\delta}$ for all $k \in \mathbb{N}$.

Proof: Take any $k \in \mathbb{N}$. We define $\bar{x}_{k+1}$ and $\bar{y}_{k+1}$ to be the state and output of a linear differential process with the augmented state matrix $\bar{\Phi}$ of the linearization, with $\bar{x}_{k+1}(0) = x_{k+1}(0)$ and $\bar{y}_k = y_k$. Since the $\mathcal{L}_1$ norm of an LTI system is its induced $\mathcal{L}_\infty$ norm, and the initial condition will decay exponentially, it is straightforward to check that $\|\bar{y}_{k+1}\|_{\mathcal{L}_\infty} \leq \mu \|y_k\|_{\mathcal{L}_\infty} + \nu \|x_{k+1}(0)\|_\infty$ for some positive ν.

Now using (7), the output error between the linear and nonlinear systems, $\tilde{y}_{k+1}(t) \triangleq y_{k+1}(t) - \bar{y}_{k+1}(t)$, is given by

$$\tilde{y}_{k+1}(t) = C \int_0^t e^{A(t-\tau)} b(x_{k+1}(\tau), y_k(\tau))\, d\tau + d(x_{k+1}(t), y_k(t)), \quad \forall t \in [0, T].$$

The convolution integral above is an LTI system with realization $(A, I, C, 0)$ and zero initial conditions, with b as its input. As A is stable, this means that the integral operator will have finite $\mathcal{L}_1$ norm. Note that both inputs are bounded by virtue of the piecewise continuity assumption on the signals and the continuity of φ, since we are on a compact interval. Therefore, we have $\|\tilde{y}_{k+1}\|_{\mathcal{L}_\infty} \leq \eta \|\varphi_{k+1}\|_{\mathcal{L}_\infty}$ for some positive η, where $\varphi_{k+1}(t) = \varphi(x_{k+1}(t), y_k(t))$ with a slight abuse of notation. Consequently, application of the triangle inequality on y_{k+1}, and the bound on $x_{k+1}(0)$, gives us $\|y_{k+1}\|_{\mathcal{L}_\infty} \leq \mu \|y_k\|_{\mathcal{L}_\infty} + \nu \bar{\kappa}_x + \eta \|\varphi_{k+1}\|_{\mathcal{L}_\infty}$. Similarly, from lemma 2, there exist positive constants α, β such that $\|x_{k+1}\|_{\mathcal{L}_\infty} \leq \alpha \|y_k\|_{\mathcal{L}_\infty} + \beta \bar{\kappa}_x$ holds in a neighborhood $Z \subseteq \mathbb{R} \times \mathcal{L}_\infty$ of the origin; i.e. for some $\delta_z > 0$ such that $\|y_k\|_{\mathcal{L}_\infty} < \delta_z$ and $\|x_{k+1}\|_\infty < \bar{\kappa}_x < \delta_z$.

Now recall that $\|\varphi(z)\|$ is $o(\|z\|)$ as $z \to 0$, for any given norm $\|.\|$. Let $\epsilon \triangleq ((1-\mu)/(3\eta)) \min\{1, \alpha^{-1}\}$. Then there exists $\delta > 0$ such that if $\|z\|_\infty < \delta$, then we have $\|\varphi(z)\|_\infty \leq \epsilon \|z\|_\infty$. We choose any $\bar{\delta}_x < \min\{\delta, \delta_z\}$ positive, and let $\bar{\delta}_y \triangleq (\bar{\delta}_x/2) \min\{1, \alpha^{-1}\}$. We claim that if

$$\bar{\kappa}_x \in \left(0, \min\left\{\bar{\delta}_x, \frac{\bar{\delta}_x - \alpha\bar{\delta}_y}{\beta}, \frac{(1-\mu)\bar{\delta}_y - \eta\epsilon\bar{\delta}_x}{\nu}\right\}\right), \quad (8)$$

and $\|y_k\|_{\mathcal{L}_\infty} < \bar{\delta}_y$, wherein $\bar{\kappa}_x < \bar{\delta}_x$ ensures that the boundary satisfies $\|x_{k+1}(0)\|_\infty < \bar{\delta}_x$, and the reader can verify that the set is nonempty, then

$$\|x_{k+1}\|_{\mathcal{L}_\infty} < \alpha\bar{\delta}_y + \beta\bar{\kappa}_x < \bar{\delta}_x,$$
$$\|y_{k+1}\|_{\mathcal{L}_\infty} < \mu\bar{\delta}_y + \nu\bar{\kappa}_x + \eta\epsilon\bar{\delta}_x < \bar{\delta}_y.$$

Indeed, if the above conditions are met, $(x_{k+1}(0), y_k)$ is in Z, the region of the space where small signal finite gain stability holds, and the first inequality can be shown to be valid via lemma 2 and routine manipulations. Then, since $\|x_{k+1}\|_{\mathcal{L}_\infty} < \delta_x$ and $\|y_k\|_{\mathcal{L}_\infty} < \delta_x$ with $\delta_x < \delta$, the trajectory $(x_{k+1}(t), y_k(t))$ is constrained to a region where $\|\varphi(x_{k+1}(t), y_k(t))\|_\infty \leq \epsilon\delta$ for all $t \in [0, T]$, and the second inequality can also be shown to be true. Hence we have found a neighborhood of the origin in $\mathcal{L}_\infty \times \mathcal{L}_\infty$ that is invariant in the pass domain. It follows by induction that if we have $\|y_0\|_{\mathcal{L}_\infty} < \bar{\delta}_y$ and $\|x_{k+1}\|_\infty < \bar{\kappa}_x$ for all $k \in \mathbb{N}$, for any $\bar{\kappa}_x$ satisfying (8), then $\|y_k\|_{\mathcal{L}_\infty} < \bar{\delta}_y$ and $\|x_{k+1}\|_{\mathcal{L}_\infty} < \bar{\delta}_x$ for all $k \in \mathbb{N}$.

Since our choice of $\bar{\delta}_x$ was arbitrary up to an upper bound, setting $\bar{\delta} = \bar{\delta}_x$ (as $\bar{\delta}_y < \bar{\delta}_x$) completes the proof. ■

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