

Iterative Learning Control of Iteration Varying Systems via Robust Update Laws with Experimental Implementation

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Abstract

Iterative learning control (ILC) is an efficient way of improving the tracking performance of repetitive systems. While ILC can offer significant improvement to the transient response of complex dynamical systems, the fundamental assumption of iteration invariance of the process limits potential applications. Utilizing abstract Banach spaces as our problem setting, we develop a general approach that is applicable to the various frameworks encountered in ILC. Our main result is that robust invariant update laws lead to stable behavior in ILC systems, where iteration varying systems converge to bounded neighborhoods of their nominal counterparts when uncertainties are bounded. Furthermore, if the uncertainties are convergent along the iteration axis, convergence to the nominal case can be guaranteed.

Keywords: Recursive control algorithms, learning control, iterative methods, robustness, uncertainty.

1. Introduction

Iterative learning control (ILC) has been recognized as an efficient way of improving the tracking performance of repetitive systems since the early 1980s (Arimoto et al., 1984). ILC can offer significant improvement to the transient response of complex dynamical systems with a high level of uncertainty through relatively simple algorithms (Bristow et al., 2006; Moore, 1993). The fundamental assumption that enables the success of these algorithms has been iteration invariance of the: 1) plant dynamics, 2) exogenous disturbances, 3) initial conditions, and 4) reference signals. This assumption greatly simplifies the ILC problem and enables the control engineer to design an asymptotically stable recurrence relation in the iteration domain by employing a contraction mapping. Even though the assumption is unrealistic, similar to feedback control of linear time invariant (LTI) systems, it yields good results in practice provided that the variation of the process (dynamics, exogenous disturbances, initial conditions etc.) from trial to trial is small.

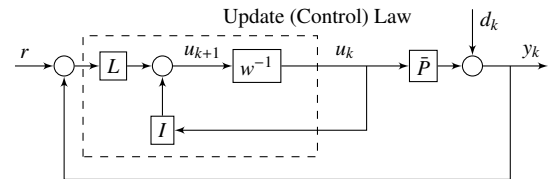


Figure 1: Feedback control in the iteration domain interpretation of ILC: The integral action of the control law ensures that the output y_k converges to r for constant $d_k = d$ for the static plant \bar{P} , provided the feedback loop is stable. Here, w^{-1} represents the trial delay operator, and d_k is a term that represents disturbances and the effect of initial conditions.

1.1. The Feedback Analogy

The restrictive nature of the invariance assumption is perhaps best understood via an analogy to feedback control, since a common interpretation of ILC is that of a feedback controller in the iteration domain, as per the following discussion: Let $\bar{P} : U \rightarrow Y$ be a bounded linear operator, where U is the space of admissible inputs and Y is the space of outputs. Assuming that \bar{P} is known and there are no exogenous signals apart from u_k affecting the output, the classical ILC problem can be stated as that of finding a controller C that maps the input history $u_0, u_1, \dots, u_{k-1} \in U$ to the current input u_k , such that the output $y_k = \bar{P}u_k$ converges to a desired reference r in the image of \bar{P} as $k \rightarrow \infty$. In most cases, C

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is designed to consider only the previous iteration, thus giving rise to the name first order ILC. The internal model principle then dictates that the controller (update law) C includes integral action to guarantee perfect tracking in the limit, so $C(u_k) = u_{k-1} + L(r - \bar{P}u_{k-1})$, as can be seen in Figure 1, which guarantees $y_k \rightarrow r$ even in the case where the output is corrupted by a constant vector $d \in Y$ such that $y_k = \bar{P}u_k + d$. Essentially, the ILC problem is that of designing a “time” invariant feedback controller for a constant static plant to track step references (Moore, 1993), under the assumption of constant disturbance signals.

The objective of this paper is to generalize the ILC problem by relaxing the invariance assumption, which restricts the feedback analogy to setpoint tracking, and fails to capture the generality associated with the feedback paradigm. In practice, initial conditions and disturbances are *always subject to variations*, while references and plants can commonly appear as outputs of higher order internal models (HOIMs)¹ in the context of robotic manipulators doing different tasks, or freeway traffic models (Hou et al., 2012).

1.2. Literature Review

Linear feedback control encompasses a wide array of problems and their accompanying solutions, such as stabilization, robustness, optimality, sensitivity reduction, fundamental limitations, and design trade-offs. Since the 1990s, there has been an increased effort in the ILC community to generalize the classical problem in these directions. These include the synthesis of 1) robust ILC algorithms (Norrlöf, 2004; Ahn et al., 2007b; van de Wijdeven et al., 2009; Bristow, 2010; Moon et al., 1998; De Roover and Bosgra, 2000; Altin and Barton, 2014), 2) norm optimal ILC algorithms with quadratic cost functions, 3) adaptive ILC (AILC) methodologies (French et al., 1999; Tayebi, 2006; Tian and Yu, 2003; Wang et al., 2004), along with the study of performance guidelines and design trade-offs (Ahn et al., 2007b; Moore and Lashhab, 2010; Pipeleers and Moore, 2012). See also Bristow et al. (2006); Ahn et al. (2007a); Xu (2011) and the references therein.

Implicit in the vast majority of these earlier works is the invariance assumption in some form. To date, there has been relatively limited material attempting to relax these assumptions. Among these, initial condition invariance was by far the most discussed topic earlier

in the literature, since perfect resetting can be hard to achieve for certain systems (Heinzinger et al., 1992). The central result of Heinzinger et al. (1992) shows that initial condition resetting errors and bounded disturbances affect the tracking error continuously, provided they are uniformly bounded in the iteration domain. The effects of varying disturbance signals have been studied in stochastic settings (Bristow, 2010; Norrlöf, 2004; Ahn et al., 2007b; Saab, 2006). Varying references are also increasingly studied in ILC theory; AILC is one of the avenues in which this objective is pursued (Xu and Xu, 2004; Xu, 2011), while some other works consider parametrizing the set of references by basis functions (Hoelzle et al., 2011; Bolder and Oomen, 2015; Bolder et al., 2014; van Zundert et al., 2016) or library based interpolations (Hoelzle and Barton, 2012). Lastly, iteration varying plant models are actively studied in the case that they can be described by a HOIM (Yin et al., 2010), with generalizations to iteration varying references and signals considered in Zhu et al. (2015).

Despite all these efforts, the feedback interpretation of ILC still paints mostly an incomplete picture, and lacks the fundamental notions of *asymptotic and input-output stability*. In this sense, the introduction of the w transform (z transform in the iteration domain) in Chen and Moore (2002) has been crucial in adopting a more holistic view of ILC as an input-output system, induced by feedback control in the iteration domain. The transform enables the integration of iteration varying signals into the ILC problem and is a good step towards the establishment of input-output stability properties in ILC. However, it restricts the analysis to iteration invariant plants and update laws. On the other hand, while Norrlöf and Gunnarsson (2002) presents a framework to investigate the stability of discrete time iteration varying systems, the analysis is restricted to iteration invariant signals. Finally, a robust ILC framework for discrete time systems in state space form is analyzed recently in Meng and Moore (2016, 2014), wherein the treatment is limited to classical D-type ILC algorithms. While the results of these two papers are theoretically important, the authors make no comments on how the learning gain matrices can be designed when the sole information on the uncertainty is boundedness.

Our aim in this paper is to construct a *general framework encapsulating a broad class of systems* in order to, 1) analyze stability properties of ILC in the presence of iteration varying signals (including references) and plant operators, where the operators are assumed to belong to a bounded set and otherwise unknown, and 2) *connect our analysis to the robust ILC literature* by showing that robust updates lead to stable behavior in

¹That is, systems wherein the plant operator P_k at trial k is a function of $P_{k-1}, P_{k-2}, \dots, P_{k-n}$ for some n . However, to the best of our knowledge, there have been no studies on whether HOIMs occur naturally in physical systems.

ILC. In addition, we will compare the performance of this uncertain iteration varying system to its nominal invariant counterpart, discuss how nominal performance can be recovered, and verify the theory with simulation examples and experimental implementation.

1.3. Organization of the Paper

The remainder of the manuscript is organized as follows: Section 2 introduces preliminaries and the ILC problem. Section 3 proves the basic boundedness result of the algorithm. In Section 4, asymptotic performance and design trade-offs are investigated. Section 5 describes the experimental setup, which also forms the basis for the simulation examples. Simulation examples are presented in Section 6, with the experimental results following in Section 7. Finally, concluding remarks are given in Section 8.

2. Background and Problem Statement

Consider the classical first order ILC problem discussed in Section 1. We assume U and Y to be Banach spaces equipped with suitable norms. We base this assumption on the fact that Banach spaces are the natural settings of contraction mapping based ILC, which relies on the Banach fixed point theorem. Furthermore, \mathcal{L}_p and l_p spaces, the natural framework for one dimensional dynamic systems, are complete. The motivation for this assumption is to come up with a general framework that contains the variety of different settings in ILC, consistent with the vector space approach in Moore (1993).

The Banach space framework is discussed further in Appendix A. For simplicity, the reader can assume \bar{P} to be an appropriate real lower triangular (causal) matrix describing a discrete time linear system, or a stable transfer function $\bar{P}(s)$, without any loss of generality.

2.1. Notation and Preliminaries

We take \mathbb{N} to represent the set of nonnegative integers and \mathbb{N}^+ the set of positive integers. For normed vector spaces X and V , $B(X, V)$ is the space of all bounded linear operators from X to V . We use $\|\cdot\|$ to denote vector and induced operator norms in the relevant spaces. For a family of operators indexed by a subset of \mathbb{N} , the product notation indicates the composition of the operators in increasing order; e.g. $\prod_{i=j}^k H_i \triangleq H_k H_{k-1} \dots H_j$ for $j \leq k$ and $\prod_{i=j}^k H_i \triangleq I$ for $j > k$, where I is the identity. The uniform distribution over $[a, b]$ is denoted $\mathcal{U}(a, b)$.

For a rigorous study of the convergence and stability properties of the iterative problem, we define the spaces² $U^\omega \triangleq \prod_{k \in \mathbb{N}} U$ and $Y^\omega \triangleq \prod_{k \in \mathbb{N}} Y$. An element x in these spaces will be defined so x_k denotes the k th coordinate. We will use this notation to refer to any sequence of objects in the same space, e.g. $x \triangleq (x_0, x_1, \dots)$ where each x_k can be an element of U , Y , or an operator in these spaces. In addition, we introduce the following definitions where the spaces X and V are in $\{U, Y\}$.

Definition 1. Let x be an element of X^ω . The norm of x is given by $\|x\| \triangleq \sup_{k \in \mathbb{N}} \|x_k\|$; x is said to be bounded if $\|x\|$ is finite.

Definition 2. A linear mapping $H : X^\omega \rightarrow V^\omega$ is *bounded-input bounded-output (BIBO) stable* if there exist a finite constant ϵ such that

$$\|(Hx)_k\| \leq \epsilon \|(x)_k\|, \quad \forall x \in X^\omega, \forall k \in \mathbb{N},$$

where $(x)_k \triangleq (x_0, x_1, \dots, x_k, 0, 0, \dots)$ is the truncation of x .

Definition 3. Let $x, v \in X^\omega$. We say x *converges to* v if $\lim_{k \rightarrow \infty} \|x_k - v_k\| = 0$. Otherwise, if

$$\limsup_{k \rightarrow \infty} \|x_k - v_k\| < \infty,$$

we say x *converges to a bounded neighborhood of* v .

Definition 4. Let $H_k \in B(X, X)$. The system defined by the equality $x_{k+1} = H_k x_k$ for all $k \in \mathbb{N}$ is *asymptotically stable* if there exists a scalar ϵ such that $\|x\| \leq \epsilon \|x_0\|$, and x converges to 0 for all $x_0 \in X$.

The framework described above will enable us to adopt a *holistic signal space approach to ILC*, with the closed loop system (in the iteration domain) as the input-output operator, so stability and convergence can be studied for the case of iteration varying factors.

2.2. System Dynamics

Based on the above, we consider the following class of systems:

$$y_k = P_k u_k + d_k, \quad \forall k \in \mathbb{N}, \quad (1)$$

where $y_k \in Y$ is the output, $u_k \in U$ is the input, $d_k \in Y$ is the exogenous signal that includes disturbances and the effect of initial conditions, and P_k is the iteration varying linear input-output operator. Moreover, we assume that each P_k is in the vicinity of the known bounded linear operator \bar{P} as stated in the following assumption.

²The notation \mathbb{R}^ω typically stands for the product $\mathbb{R} \times \mathbb{R} \times \dots$, hence U^ω, Y^ω .

Assumption 1. The input-output operators lie in a neighborhood of \bar{P} . In other words, there exists a finite real constant ρ such that

$$P_k \in \mathcal{P} \triangleq \{H \in B(U, Y) : \|H - \bar{P}\| < \rho\}, \quad \forall k \in \mathbb{N}.$$

Due to the assumption that the process variables P_k and d_k are varying along the iteration axis, it is a straightforward matter to assume that the reference is also subject to variations from trial to trial. Thus, our objective is to solve the following problem:

Problem 1. Find an ILC update law such that the error vector e defined by $e_k \triangleq r_k - y_k$ for all $k \in \mathbb{N}$, where the reference r_k is in the image of \bar{P} for all $k \in \mathbb{N}$, converges to a small neighborhood of 0.

As with the plant operators, we make a boundedness assumption on r .

Assumption 2. The reference vectors lie in a neighborhood of a nominal reference \bar{r} in the image of \bar{P} . In other words, there exists a finite real constant ζ such that

$$r_k \in \mathcal{R} \triangleq \{h \in \bar{P}(U) \subset Y : \|h - \bar{r}\| < \zeta\}, \quad \forall k \in \mathbb{N}.$$

3. Stability of Iteration Varying Systems with Robust Update Laws

This section will detail the stability analysis of our proposed solution to Problem 1. The solution will generalize the findings of Norrlöf and Gunnarsson (2002) along the abstract contraction mapping approach of Moore (1993), and connect the iteration varying problem to the robust ILC literature. Consider the most general linear iteration invariant update law

$$u_{k+1} = Qu_k + Le_k, \quad \forall k \in \mathbb{N}, \quad (2)$$

where Q and L are bounded, and u_0 is arbitrary. Furthermore, we will require the update law to be subject to the robustness condition

$$\|Q - LH\| \leq \gamma < 1, \quad \forall H \in \mathcal{P}, \quad (3)$$

for some real constant γ , which guarantees monotonic convergence for all $H \in \mathcal{P}$ when the system is iteration invariant.

Remark 1. As opposed to Meng and Moore (2016), we consider only iteration invariant update laws, which enables us to connect our analysis to the robust ILC literature. While the ensuing analysis holds for iteration varying update laws, it is not yet clear how such iteration varying update laws may be designed since the plant variations considered in our work are unknown.

Condition (3) is a sufficient condition for asymptotic stability of the iteration varying input equation, as we shall see below. When the spaces U and Y are finite dimensional, i.e. Q , L , and all $H \in \mathcal{P}$ have matrix representations, (uniform/robust) asymptotic stability is equivalent to the *joint spectral radius* of the bounded set of operators $(Q - LP)$ being strictly less than 1, which is conjectured to be an undecidable problem (Blondel et al., 2004).

Substituting (1) into the update law (2) yields the recurrence relation

$$u_{k+1} = T_k u_k + L\eta_k, \quad \forall k \in \mathbb{N}, \quad (4)$$

where $T_k \triangleq Q - LP_k$ and $\eta_k \triangleq r_k - d_k$. The solution of the input vector in terms of u_0 and η_k can then be given as

$$u_{k+1} = \underbrace{\left(\prod_{i=0}^k T_i\right)}_{\text{Natural response}} u_0 + \sum_{i=0}^k \underbrace{\left(\prod_{j=i+1}^k T_j\right)}_{\text{Forced response}} L\eta_i, \quad \forall k \in \mathbb{N}. \quad (5)$$

Equation (4) defines a “time” (iteration) varying discrete dynamical system on the space U . As such, its solution (5) is conceptually the same as that of a discrete time system on \mathbb{R}^n . When (3) holds, since $\|T_k\| \leq \gamma < 1$, it is easy to see that (4) is a well-defined, stable dynamical system.

Proposition 1. *The linear iterative system described by (4) with $\eta = 0$, subject to (3), is asymptotically stable.*

Proof. Assume (3) holds and $\eta = 0$. Take any $u_0 \in U$. Then from (5), $\|u_{k+1}\| \leq \gamma^{k+1} \|u_0\|$. Since $\gamma < 1$, it follows that u converges to 0 and $\|u\| \leq \|u_0\|$. Therefore, system (4) is asymptotically stable. \square

Proposition 2. *The linear iterative system described by (4) with input η , subject to (3) and the equality $u_0 = 0$, is BIBO stable.*

Proof. Assume (3) holds and $u_0 = 0$. Take any $\eta \in Y^\omega$. Then from (5) we have

$$\begin{aligned} \|u_{k+1}\| &\leq \sum_{i=0}^k \gamma^{k-i} \|L\| \|\eta\|_k = \frac{1 - \gamma^{k+1}}{1 - \gamma} \|L\| \|\eta\|_k \\ &\leq \frac{\|L\| \|\eta\|_k}{1 - \gamma} \leq \frac{\|L\| \|\eta\|_{k+1}}{1 - \gamma}, \quad \forall k \in \mathbb{N}, \end{aligned}$$

where we use the fact that the truncated norm is monotonically increasing by definition. Using the same property, we can show by the above inequality that

$$\|(u)_k\| = \max_{i=1, \dots, k} \|u_i\| \leq \frac{\|L\| \|\eta\|_k}{1 - \gamma}, \quad \forall k \in \mathbb{N}. \quad (6)$$

Therefore, system (4) is BIBO stable. \square

We showed that the recursion relation (4) is asymptotically and BIBO stable when subject to (3). We finish this section with the following theorem, which shows that u and y are bounded if d is bounded.

Theorem 1. *The signals u and y of the linear iterative system (1) with the update law (2) is bounded if d is bounded.*

Proof. Consider the solution (5) of the input u , which is the superposition of the *natural response* describing the asymptotic response to the initial condition u_0 , and the *forced response* describing the input-output behavior due to η . Since r is bounded by Assumption 2, η is bounded if d is bounded. From Propositions 1 and 2, it follows that u is bounded. Now observe that

$$\|y_k\| \leq \|P_k\| \|u_k\| + \|d_k\| \leq \|P_k\| \|u\| + \|d\|, \quad \forall k \in \mathbb{N},$$

by (1). Since the set \mathcal{P} is bounded, it follows that y is bounded. \square

The results of this section show that an ILC update law can be safely applied on iteration varying systems, provided the update law is designed to be robust against plant uncertainties. Based on the nature of the underlying spaces U, Y , and the operator set \mathcal{P} , this update law can be designed using existing robust ILC techniques.

4. Asymptotic Performance and Design Trade-offs

Having shown that the ILC system with our proposed solution is well-posed under the robustness assumption, we will direct our attention to the asymptotic performance of the system, when compared to a nominal iteration invariant system. One motivation for analyzing these systems in general, as opposed to systems where $Q = I$, is that perfect tracking can be an infeasible objective for various reasons. For example, the set \mathcal{P} might be too big, so (3) cannot be satisfied for $Q = I$. As such, we will introduce a nominal iterative system via the known operator \bar{P} and reference \bar{r} under the assumption that $d = 0$, which will facilitate our analysis. The results of this section will be stated without proof to keep the discussion at a high level; interested readers are referred to Altin and Barton (2015) for detailed explanations.

4.1. Asymptotic Response of the System and the Corresponding Nominal Dynamics

As the choice of u_0 has no effect on the input (5) as the iteration index $k \rightarrow \infty$, we will drop the natural re-

sponse from (5), and consider

$$\begin{aligned} u_{k+1} &\triangleq \sum_{i=0}^k \left(\prod_{j=i+1}^k T_j \right) L \eta_i, \\ e_k &\triangleq -P_k u_k + \eta_k, \end{aligned} \quad (7)$$

for all $k \in \mathbb{N}$, where $u_0 = 0$.

We define the nominal asymptotic system to be the case where the signal $d = 0$ and the plant $P_k = \bar{P}$ for all $k \in \mathbb{N}$. In other words, we describe the nominal system as

$$\bar{y}_k = \bar{P} \bar{u}_k, \quad \forall k \in \mathbb{N},$$

where $\bar{y}_k \in Y$ is the nominal output and $\bar{u}_k \in U$ is the nominal input. Thus, the error dynamics of the nominal system are given by the relation below, where $\bar{\eta} \triangleq \bar{r}$:

$$\bar{e}_k = -\bar{P} \bar{u}_k + \bar{\eta}, \quad \forall k \in \mathbb{N}.$$

We take the update law as $\bar{u}_{k+1} = Q \bar{u}_k + L \bar{e}_k$, with Q and L the same as before. Consequently, since the choice of \bar{u}_0 has no effect in the limit, we consider

$$\begin{aligned} \bar{u}_{k+1} &\triangleq \sum_{i=0}^k \left(\prod_{j=i+1}^k \bar{T} \right) L \bar{\eta}, \\ \bar{e}_k &\triangleq -\bar{P} \bar{u}_k + \bar{\eta}, \end{aligned} \quad (8)$$

for all $k \in \mathbb{N}$, where $\bar{T} \triangleq Q - L\bar{P}$ and $\bar{u}_0 = 0$. This nominal system is well known to be stable and convergent, with the limits $\bar{u}_\infty \triangleq \lim_{k \rightarrow \infty} \bar{u}_k$ and $\bar{e}_\infty \triangleq \lim_{k \rightarrow \infty} \bar{e}_k$, when (3) holds.

4.2. Asymptotic Learning Performance

We will now analyze the performance of the algorithm (2) on the ILC system. Towards that end, based on the results of the previous section, we will compare the dynamics (7) and (8) written below in recursive form:

$$\bar{u}_{k+1} = \bar{T} \bar{u}_k + L \bar{\eta}, \quad \forall k \in \mathbb{N}, \quad (9)$$

$$u_{k+1} = T_k u_k + L \eta_k, \quad \forall k \in \mathbb{N}. \quad (10)$$

The equalities above will enable us to show that the iteration varying ILC system converges to a bounded neighborhood of the nominal invariant system. In showing this result, the main idea is to subtract the system (10) from the nominal dynamics (9) and come up with a stable recursion, driven by the bounded uncertainties due to P, r, d .

Theorem 2. *Assume that the linear iterative system described by (1) with the update law (2) is subject to (3).*

Then, if d is bounded, u and e converge to a neighborhood of \bar{u} and \bar{e} , respectively. In other words,

$$\limsup_{k \rightarrow \infty} \|\tilde{u}_k\| \leq \|L\| \frac{\rho \|\bar{u}_\infty\| + \zeta + \|d\|}{1 - \gamma}, \quad (11)$$

and

$$\limsup_{k \rightarrow \infty} \|\tilde{e}_k\| \leq \left(\|L\| \frac{\|\bar{P}\| + \rho}{1 - \gamma} + 1 \right) \times (\rho \|\bar{u}_\infty\| + \zeta + \|d\|). \quad (12)$$

In addition, if the input-output operator and the reference converge to the nominal case, and d converges to 0, it can be shown that the ILC system converges to the nominal invariant system, as discussed in the following theorem. Here, convergence of P to \bar{P} is to be interpreted as $\lim_{k \rightarrow \infty} \|P_k - \bar{P}\| = 0$ as in Definition 3.

Theorem 3. *Assume that the linear iterative system described by (1) with the update law (2) is subject to (3). Then, if P converges to \bar{P} , r converges to \bar{r} , and d converges to 0, u and e converge to \bar{u} and \bar{e} , respectively.*

Theorems 2 and 3 are significant results for the following reasons: First, the bounds in (11) and (12) are continuous increasing functions of the uncertainties quantified by the scalars ρ , ζ , and the disturbance magnitude $\|d\|$. As such, decreased levels of uncertainty imply that system response can be guaranteed to be closer to its nominal counterpart. Moreover, in the case where $\rho = \zeta = 0$ and $d = 0$, (11) and (12) predict that the asymptotic response is equal to that of the nominal system, as expected. Second, in the case that the uncertainties vanish asymptotically, we can guarantee that the nominal response can be recovered in the limit.

4.3. Design Trade-offs

As in the iteration invariant case, it is trivial to show that γ is a measure of the convergence speed³ of the algorithm: Recall from Section 1 that the input and error converge to the forced response of the ILC system. Furthermore, we saw in Section 3 that the effect of the initial input vanishes geometrically with rate γ . Hence, lower values of γ correspond to faster convergence to the forced response of the system, and vice versa.

Let $\alpha \triangleq \|L\|/(1 - \gamma)$. We note that from (6), the bound $\|\bar{u}_\infty\| \leq \alpha \|r\|$ can be derived for the nominal case.

³More strictly, the convergence speed of the algorithm would be the smallest γ satisfying (3).

Plugging this into (11) and (12), without loss of generality, it is easy to see that both the input and output asymptotic errors ($\limsup_{k \rightarrow \infty} \|\tilde{u}_k\|$ and $\limsup_{k \rightarrow \infty} \|\tilde{e}_k\|$) decrease as α decreases. Moreover

$$\lim_{\alpha \rightarrow 0} \left(\limsup_{k \rightarrow \infty} \|\tilde{u}_k\| \right) = 0,$$

and

$$\limsup_{\alpha \rightarrow 0} \left(\limsup_{k \rightarrow \infty} \|\tilde{e}_k\| \right) \leq \zeta + \|d\|,$$

since $\lim_{\alpha \rightarrow 0} \|\bar{u}_\infty\| = 0$ by (6). However, we note that decreasing α might come at the expense of steady state performance. In the simulation examples and experimental implementation, we will use this fact to design optimal algorithms given steady state performance constraints.

4.4. Constrained Optimal Design for Predictable Performance

By definition of α , the ILC problem can be formulated as a constrained minimization of the following form:

$$\begin{aligned} & \underset{\substack{Q \in B(U,U) \\ L \in B(Y,U)}}}{\text{minimize}} && \frac{\|L\|}{1 - \bar{\gamma}} \\ & \text{subject to} && \bar{\gamma} = \|Q - L\bar{P}\| + \rho \|L\| \leq \sigma < 1, \\ & && \|I - \bar{P}(I - Q + L\bar{P})^{-1}L\| \leq \beta, \end{aligned} \quad (13)$$

for some $\sigma \in [0, 1)$ and $\beta \in (0, \infty)$. In (13), the constraint $\|Q - L\bar{P}\| + \rho \|L\| \leq \sigma < 1$ is the robust stability criterion derived by applying the triangle inequality on the uncertainty set \mathcal{P} described by Assumption 1. This constraint can be relaxed as

$$\sup_{H \in \mathcal{P}} \|Q - LH\| \leq \sigma < 1,$$

at the expense of computational complexity. On the other hand, the constraint $\|I - \bar{P}(I - Q + L\bar{P})^{-1}L\| \leq \beta$ sets a limit on the allowable nominal steady state error \bar{e}_∞ since

$$\bar{y}_\infty = \bar{P}(I - (Q - L\bar{P}))^{-1}L\bar{r}, \quad (14)$$

and therefore

$$\bar{e}_\infty = (I - \bar{P}(I - Q + L\bar{P})^{-1}L)\bar{r}.$$

Thus, the objective of the nonlinear program (13) is to find a robust linear ILC update law with guaranteed nominal steady state performance, that minimizes the deviations from the nominal system. The program (13) will be solved numerically via the MATLAB command `fmincon` and verified via simulations and experiments in the following sections.

5. Description of the Experimental Setup

This section describes the experimental setup that will be used to verify the findings of the previous sections. We will be working with single-input single-output (SISO) discrete time linear dynamic systems over a fixed finite horizon, i.e. the spaces $U = Y = \mathbb{R}^n$ for some positive integer n , equipped with the 2 norm. Hence, given any $x \in (\mathbb{R}^n)^\omega = \mathbb{R}^n \times \mathbb{R}^n \times \dots$, we have

$$\|x\| \triangleq \sup_{k \in \mathbb{N}} \|x_k\|_2,$$

where $\|\cdot\|_2$ denotes the 2 norm. Note that since all norms in \mathbb{R}^n are equivalent, it suffices to pick any norm satisfying the robustness condition (3) in order to conclude stability. Although the maximum norm is a more natural choice for \mathbb{R}^n due to Definition 1, we choose the 2 norm for practical purposes since the root mean squared error is the metric of interest for many applications, e.g manufacturing.

The plant set \mathcal{P} is a bounded set of $n \times n$ lower triangular (causal) nonsingular matrices. Similarly, the learning operators Q and L are $n \times n$ real matrices. Here, the inherent delay of the plant is ignored by shifting the output (Bristow et al., 2006). For example, if the system has relative degree 1, we consider the matrix equation $y_k = \bar{P}u_k$, where

$$\begin{aligned} u_k &\triangleq [u_k(0) \quad u_k(1) \quad \dots \quad u_k(n-1)]^T, \\ y_k &\triangleq [y_k(1) \quad y_k(2) \quad \dots \quad y_k(n)]^T. \end{aligned} \quad (15)$$

Similarly, the reference vector is given as

$$r_k \triangleq [r_k(1) \quad r_k(2) \quad \dots \quad r_k(n)]^T. \quad (16)$$

5.1. Plant Description

The experimental setup considered in our work is an Aerotech ALS 25010, a low profile high accuracy linear motion stage, controlled through dSPACE. The specifications of the stage (the Y stage) are detailed in Table 1. The stage is mounted onto a similar Aerotech stage (the X stage), which in turn is connected to a 600×900 mm TMC breadboard. The motion ranges of the two stages are orthogonal to each other in Cartesian coordinates, thereby forming a dual axis XY type motion control platform. For simplicity, the latter of the stages is stabilized at a fixed position by a proportional-integral-derivative (PID) controller, and the overall setup is treated as a single axis motion stage.

⁴The travel speed and linear acceleration are limited to 300 mm/s and 3 m/s², respectively, by the software.

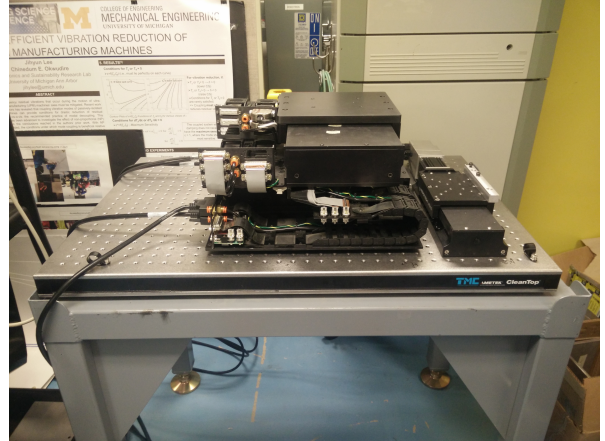


Figure 2: The experimental setup.

Table 1: Specifications of Aerotech ALS 25010⁴

Total Travel	100 mm
Servomotor	Brushless Linear
Encoder	Noncontact Linear
Resolution	0.001-0.2 μm
Maximum Travel Speed	2 m/s
Maximum Linear Acceleration	30 m/s ²
Accuracy	$\pm 1 \mu\text{m}$

5.2. Closed Loop Identification

The Y stage is controlled by a PID controller (implemented at 1 kHz), resulting in the closed loop complementary sensitivity function

$$\begin{aligned} T_{cl}(s) &= KK_p(K_d s^2 + s + K_i) \\ &\times \frac{1}{Ms^3 + (C + KK_p K_d)s^2 + (D + KK_p)s + KK_p K_i}, \end{aligned}$$

where the controller has proportional gain $K_p = 5$, integral gain $K_i = 0.3$, and derivative gain $K_d = 3.51 \times 10^{-3}$. The function $T_{cl}(s)$ is derived by combining the PID controller and the open loop empirically identified second order model with mass $M = 1$ kg, damping coefficient $C = 55$ Ns/m, spring coefficient $D = 2.6$ N/m, and open loop gain $K = 6660$.

It is well known that arbitrary small open loop modeling errors can lead to arbitrarily large closed loop modeling errors (Albertos and Sala, 2002). The identified closed loop model $T_{cl}(s)$ is inaccurate for our purposes since ILC requires a relatively high bandwidth⁵. As such, a closed loop identification experiment at 1 kHz

⁵For SISO linear discrete time ILC, the relative degree and the

is performed in order to have an accurate impulse response of the closed loop, which can be used to construct the lower triangular Toeplitz plant matrix \bar{P} . This is done by sending a Heaviside step signal as the desired reference and differentiating the output signal. The first 200 samples of the identified impulse response are shown in Figure 3, where the signal is compared to the response $T_{cl}^{\text{disc}}(z)$ derived by discretizing $T_{cl}(s)$ at 1 kHz.

5.3. The Desired Output

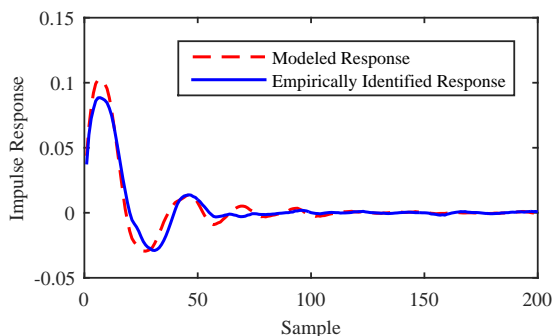


Figure 3: Modeled (via $T_{cl}^{\text{disc}}(z)$) and empirically identified closed loop impulse responses.

The used reference signal is shown in Figure 4. It is a smooth ramp up and down signal at 1 kHz and lasts for 1 s. The signal covers approximately 75 percent of the Y-stage range and sets the velocity close to the software limit so that the reference is as challenging as possible, without excessive acceleration and jerk. This is done to avoid oscillations of the base that carries the breadboard and hence uncontrollable perturbations.

5.4. Plant Perturbations

Several weights varying between 100 g and 1.5 kg are used to perturb the experimental setup: During the experiments, these weights are placed on the Y stage according to a predetermined sequence \mathcal{S} that was randomly chosen. As a result of the increased mass, the closed loop impulse response is perturbed. The magnitude of the perturbations are roughly estimated to be $\rho = 0.01$ in terms of the uncertainty description of Assumption 1.

sign of the first nonzero Markov parameter is all that is needed for a stable update law. Similarly, for SISO linear continuous time ILC, the relative degree and the sign of the corresponding feedthrough term is necessary and sufficient to design a stable update law. However, the variety of algorithms that can be used with such limited information is small, and may result in slower convergence.

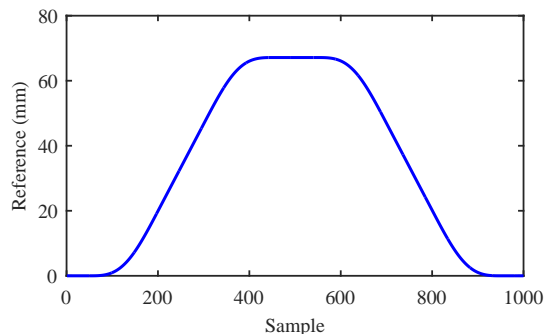


Figure 4: The desired output.

6. Simulations

As stated in Section 5, we will be working with the spaces $U = Y = \mathbb{R}^n$ for some positive integer n , equipped with the 2 norm. The plant set \mathcal{P} is composed of $n \times n$ lower triangular nonsingular matrices, and the nominal plant \bar{P} is derived from the closed loop identified impulse response (solid blue line) shown in Figure 3, unless otherwise stated. The objective of this section is twofold. First, the input-output stability of several well-known ILC algorithms under iteration varying uncertainties will be verified via simulation. Second, for certain classes of update laws, we will attempt to minimize the bounds on $\limsup_{k \rightarrow \infty} \|\tilde{e}_k\|$ using the nonlinear program (13) to obtain more predictable performance.

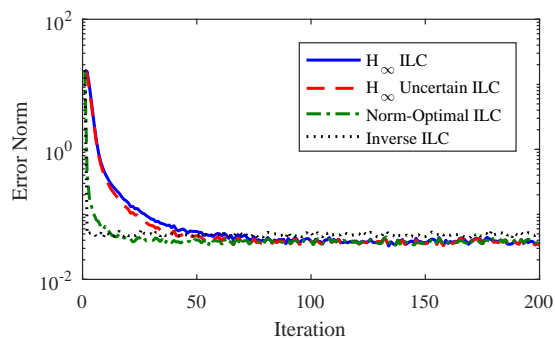


Figure 5: Comparison of several first and higher order algorithms under random perturbations. All algorithms maintain stability and boundedness under iteration varying disturbances and uncertainties. The higher order \mathcal{H}_∞ ILC algorithms exhibit significantly slower convergence compared to the first order algorithms. While inverse ILC converges in a single iteration, it has a higher steady state error, since it is sensitive to plant uncertainties and disturbances.

6.1. Stability under Iteration Varying Perturbations

Figure 5 compares the performance of four different ILC algorithms in the presence of trial varying un-

certainties and disturbances. The additive plant uncertainty ($P_k - \bar{P}$) is chosen to be a lower triangular random matrix, where each nonzero entry is drawn from $\mathcal{U}(-0.005, 0.005)$. Similarly, disturbances are considered to be a combination of input and output disturbances $d_k^{\text{in}}, d_k^{\text{out}}$, where each entry is drawn from $\mathcal{U}(-0.0025, 0.0025)$.

The ILC algorithms considered in this scenario are listed as follows:

1. \mathcal{H}_∞ ILC for certain systems.
2. \mathcal{H}_∞ ILC for uncertain systems.
3. Norm optimal ILC, in which the quadratic cost function J is minimized by solving for u_{k+1} without constraints;

$$J = e_{k+1}^T W_e e_{k+1} + u_{k+1}^T W_u u_{k+1} + (u_{k+1} - u_k)^T W_{\Delta u} (u_{k+1} - u_k), \quad (17)$$

where $W_e, W_u, W_{\Delta u}$ are positive (semi) definite matrices. To simplify the problem further for the norm optimal framework (17), we will assume that these weighting matrices are scalar multiples of the identity matrix, so $W_e = w_e, W_u = w_u, W_{\Delta u} = w_{\Delta u}$. The algorithm in Figure 5 is derived by setting the weighting parameters as $w_e = 1, w_u = 0$, and $w_{\Delta u} = 0.5$, which are heuristically tuned.

4. Inverse ILC, i.e. $Q = I$ and $L = \bar{P}^{-1}$. Note that the matrix \bar{P} is invertible since the plant set \mathcal{P} comprises nonsingular matrices.

The \mathcal{H}_∞ type ILC algorithms are described in detail in Ahn et al. (2007a) and in general yield higher order (up to order n for n samples) algorithms. However, it is a straightforward exercise to extend our analysis to n th order algorithms by augmenting (1); e.g. we can consider $y_k^{\text{aug}} = (y_{kn}, y_{kn+1}, \dots, y_{kn+n-1})$. The reader can see in Figure 5 that all algorithms maintain stability and boundedness under iteration varying disturbances and uncertainties. It is also worth noting that the higher order \mathcal{H}_∞ ILC algorithms exhibit significantly slower convergence compared to the first order algorithms.

Remark 2. A contraction based analysis is not obvious with the typical state augmentation

$$u_k^{\text{aug}} = (u_k, u_{k+1}, \dots, u_{k+n-1}).$$

For example, if the norm of the space U^n is taken such that $\|u_k^{\text{aug}}\| = \max_{i \in \{0, 1, \dots, n-1\}} \|u_{k+i}\|$, the induced norm of any operator mapping u_k^{aug} to u_{k+1}^{aug} will be at least 1. However, this issue can be circumvented by utilizing a weighted norm on U^n , for instance by taking the norm to be $\|u_k^{\text{aug}}\| = \max_{i \in \{0, 1, \dots, n-1\}} \lambda^{n-i} \|u_{k+i}\|$ for some $\lambda \in (0, 1)$.

6.2. Computation and Verification of Optimal Update Laws

To demonstrate the utility of the optimization approach to ILC design, the performance of different Q and L matrices computed via (13) will be compared. For each of the computed algorithms, a set of 200 trials will be conducted, and for each algorithm there exist positive integers N_0 and N_f such that disturbances and uncertainties affect the system from trial N_0 to N_f . The performance measure we would like to minimize is given as

$$\delta \triangleq \max_{k, j \in \{N_0, N_0+1, \dots, N_f\}} \|e_k - e_j\|. \quad (18)$$

The scalar quantity δ is an indirect measure of fluctuations from nominal performance, with lower values signifying better predictability with respect to the nominal system. The reason for considering this measure as opposed to $\max_{k \in \mathbb{N}} \|\bar{e}_k\|$ is consistency with the experimental validation, since the ‘‘nominal’’ system is not implementable in practice due to noise and disturbances.

The following steps are taken to enhance computational aspects of the problem:

- **Norm optimally derived filters:** The first case we consider is that the update law is derived via the norm optimal framework (17) with scalar weighting matrices, so

$$J = w_e \|e_{k+1}\|^2 + w_u \|u_{k+1}\|^2 + w_{\Delta u} \|u_{k+1} - u_k\|^2. \quad (19)$$

The solution of the norm optimal problem is given in the form of matrices Q, L such that

$$u_{k+1} = Qu_k + Le_k. \quad (20)$$

In other words, we impose the additional constraint on (13) that the matrices Q, L minimize the cost function (19) via (20).

A specific solution (Q, L) for given nonzero weightings $(w_e, w_u, w_{\Delta u})$ is invariant over the open set $\{\mu(w_e, w_u, w_{\Delta u}) : \mu \in (0, \infty)\}$. As such, the weighting w_e can be fixed so that the program (13) with the additional constraint defined above optimizes over the two scalars w_u and $w_{\Delta u}$.

- **Lower triangular Toeplitz filters:** In a similar fashion, to reduce complexity, we will also consider the case where Q and L are lower triangular Toeplitz matrices. This reduces the number of variables to be optimized from $2n^2$ to $2n$, significantly decreasing the computational burden. Despite this simplification, the program (13) is still computationally expensive for large n . For demonstration

Table 2: Simulation comparison of optimized update laws derived from (19) for different values of β . Here, $w_e = 10$ is the fixed weighting so the optimization is over w_u and $w_{\Delta u}$. The additive plant uncertainties are chosen to be lower triangular, where each nonzero entry is drawn from $\mathcal{U}(-0.005, 0.005)$. Similarly, each entry of $d_k^{\text{in}}, d_k^{\text{out}}$ is drawn from $\mathcal{U}(-0.0025, 0.0025)$. For each case, $\|\bar{e}_\infty\| = \beta$.

β	w_u	$w_{\Delta u}$	$\bar{\gamma}$	α	δ
0.9500	0.0263	0	0.0975	10.803	183.05
0.9990	1.1383	0	0.0134	1.3623	129.95
0.9999	10.000	0	0.0050	0.5025	70.29

purposes, the number of samples for this simplification will be chosen as 10, and the model used will be the discretization of the identified closed loop model $T_{cl}(s)$ sampled at 100 Hz. The considered reference signal is a 5 Hz unit amplitude sine wave. Note that since the output and the reference are shifted via (15) and (16), the matrix L represents a noncausal LTI filter when it is nonsingular: The input $u_k(i)$ at time i depends on the error $e_k(i+1)$, for all $i \in \{0, 1, \dots, n-1\}$.

We also note that similar simplifications can be made, for example, by choosing Q and L to be diagonal, or upper triangular and/or Toeplitz. As before, the additive plant uncertainties will be chosen to be lower triangular random matrices, where each nonzero entry is drawn from $\mathcal{U}(-0.005, 0.005)$. Similarly, the disturbances are considered to be a combination of input and output disturbances $d_k^{\text{in}}, d_k^{\text{out}}$, where each entry is drawn from $\mathcal{U}(-0.0025, 0.0025)$.

Remark 3. At first glance, optimizing an “optimal” learning law might seem redundant, but can be explained by analogy to linear quadratic regulation (LQR). LQR is an optimal control methodology in which a quadratic “cost” function is minimized to find an optimal state feedback law. In practice, the cost function and the associated weighting matrices are not given as the design specification for a control problem. Often, the weighting matrices are used as “tuning knobs” to properly adjust the resulting state feedback law and achieve given design specifications (e.g. maximum rise time and/or settling time, minimum disturbance rejection etc.). In this sense, our approach is similar to optimally selecting the LQR weights to minimize plant sensitivity, subject to a lower bound on convergence rate and an upper bound on steady state error under step responses, which can be done in a numerical fashion.

Table 2 compares update laws derived from (19) for different values of β , which bounds the accept-

able steady state error level. For all cases, the nominal asymptotic error turns out to have magnitude β ; i.e. $\|\bar{e}_\infty\| = \beta$. It can be seen that decreasing values of α signify a decreasing level of performance uncertainty, i.e. decreasing δ . Moreover, there seems to be a trade-off between β and δ , so predictable performance comes at the expense of nominal performance.

The norm optimal framework (19) gives limited design freedom since only two scalar variables are optimized. The usefulness of the optimization approach (13) can be seen better in Figure 6, where 10×10 lower triangular Toeplitz matrices Q and L are optimized, as noted before. To further verify the trade-off between α and δ , different lower bounds on α are set as optimization constraints, while β is kept constant. The update law with $\alpha = 0.6882$ yields more predictable performance compared to when $\alpha = 3.3345$, which can also roughly be seen from the fact that the latter achieves a higher maximal and a lower minimal error, while the nominal asymptotic performance is the same.

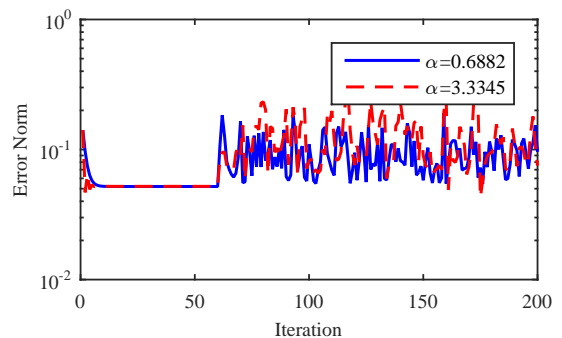


Figure 6: Performance of optimized lower triangular Toeplitz controllers with: For $\alpha = 3.3345$ we have $\delta = 8.3930$, and for $\alpha = 0.6882$ we have $\delta = 3.1896$. The additive plant uncertainties are chosen to be lower triangular, where each nonzero entry is drawn from $\mathcal{U}(-0.05, 0.05)$. Similarly, each entry of $d_k^{\text{in}}, d_k^{\text{out}}$ is drawn from $\mathcal{U}(-0.25, 0.25)$.

Remark 4. In Table 2, the optimal weighting $w_{\Delta u} = 0$ in all cases, for which an intuitive explanation can be given as follows: For iteration invariant systems and disturbances, the weight $w_{\Delta u}$ does not affect the converged error (the scalar β in (13)), and a larger $w_{\Delta u}$ leads to a slower convergence. Thus, for the objective of minimizing (13), a nonzero $w_{\Delta u}$ leads to a suboptimal solution since fast convergence is desired to minimize α . However, this might not necessarily be the case for different formulations of the nonlinear program (13), larger values of the uncertainty bound ρ , or nonscalar norm optimal ILC weightings.

7. Experimental Results

In this section, experimental implementation results for the update laws derived in Section 6 will be presented. The objectives of the section are similar to that of Section 6. That is, we would like to verify experimentally the input-output stability of a couple of ILC algorithms under iteration varying uncertainties. Second, we would like to roughly verify the optimization approach (via the nonlinear program (13)) to norm optimal ILC synthesis by comparing the experimental performance of the update laws whose simulation results are shown in Table 2. As an additional point, we will discuss the idea of *precompensation in the iteration domain* and test this idea on our experimental setup. In this section, experimental implementation results for the update laws derived in Section 6 will be presented. The objectives of the section are similar to that of Section 6. That is, we would like to verify experimentally the input-output stability of a couple of ILC algorithms under iteration varying uncertainties. Second, we would like to roughly verify the optimization approach (via the nonlinear program (13)) to norm optimal ILC synthesis by comparing the experimental performance of the update laws whose simulation results are shown in Table 2. As an additional point, we will discuss the idea of *precompensation in the iteration domain* and test this idea on our experimental setup.

7.1. Robust Stability of First and Higher Order ILC

We will compare the \mathcal{H}_∞ ILC algorithm for certain systems described in Ahn et al. (2007a) with a simple manually tuned norm optimal controller; the particular \mathcal{H}_∞ algorithm is chosen since it requires significantly less time to be synthesized and has similar performance compared to its uncertain counterpart (see Figure 5). For robustness against high frequency noise amplification, the computed input u_{k+1} of the \mathcal{H}_∞ controller is further filtered through a first order low pass filter with a cutoff frequency of 400 Hz. The norm optimal controller has the scalar weightings $w_e = 10$, $w_u = 0$, and $w_{\Delta u} = 5$. At the samples where the velocity of the reference signal is equal to 0, a first order low pass filter with cutoff frequency of 150 Hz is applied to ensure robustness against high frequency noise amplification and avoid numerical instability. The results can be seen in Figure 7, where both systems maintain stability and portray comparable performance under unknown bounded perturbations from trials 25 to 45, where the predefined sequence \mathcal{S} of weights is placed on the Y stage.

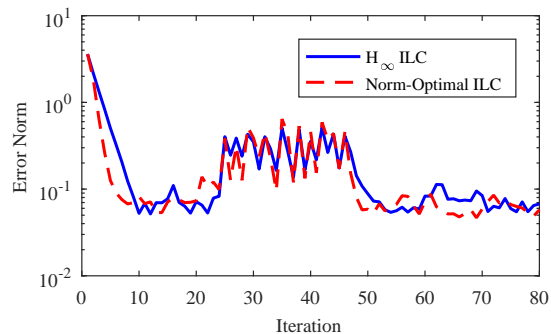


Figure 7: Comparison of \mathcal{H}_∞ and norm optimal ILC algorithms under bounded unknown perturbations from trials 25 to 45, where the predefined sequence \mathcal{S} of weights is placed on the Y stage from trials 25 to 45. Both systems maintain stability and portray comparable performance under unknown bounded perturbations.

Table 3: Experimental comparison of optimized update laws derived from (19) for different values of β . Here, $w_e = 10$ is the fixed weighting so the optimization is over w_u and $w_{\Delta u}$. For each case, $\|\bar{e}_\infty\| = \beta$. Decreasing values of α signify decreasing values of δ , as expected.

β	w_u	$w_{\Delta u}$	$\bar{\gamma}$	α	δ
0.9500	0.0263	1	0.0975	57.9360	1.2274
0.9990	1.1383	1	0.4193	1.7637	1.1830
0.9999	10.000	1	0.0909	0.9689	0.5244

7.2. Optimized Update Laws

The norm optimal controllers derived from (13), whose simulation results are shown in Table 2, are tested on the experimental setup to verify the hypothesis that δ can be minimized via the program (13). However, to avoid high frequency noise amplification, we set $w_{\Delta u} = 1$. The predefined sequence \mathcal{S} of weights is placed on the Y stage as before from trials 25 to 45. We note that we use the scalar quantity δ defined in (18), since the “nominal” system is not implementable in practice due to noise and disturbances. As can be seen in Table 3, decreasing values of α signify decreasing values of δ , which is expected. Note that δ values are much lower compared to their simulated values, which is due to the fact that the experimental perturbations are limited to several different weights as opposed to the random perturbations of the simulation scenarios.

7.3. Precompensation in the Iteration Domain

Perfect tracking is an infeasible objective when the system to be controlled is subject to unknown iteration varying disturbances and/or, when the additive uncertainty is high in magnitude. As such, depending on the magnitude of uncertainties, minimizing the measure α can be taken as an objective of primary importance over

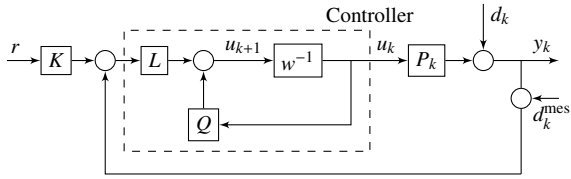


Figure 8: Precompensation in the iteration domain: The feedback control in the iteration domain interpretation of ILC makes it clear that aggressive learning might amplify measurement noise d_k^{mes} . When the controller does not have an integrator, i.e. $Q \neq I$, the precompensator K can be used to partially recover the tracking performance.

the steady state performance. This approach has not been explored much in the ILC literature. To be precise, while plenty of publications have studied how to reduce the absolute error, not much work has been done to quantify the *relative error* \tilde{e}_k in the presence of iteration varying effects. For certain applications (e.g. manufacturing), precision is arguably more important than accuracy, and repeatable errors are preferred. When this is the case and perfect tracking is infeasible or undesirable due to large uncertainties, and/or iteration varying effects, we propose *precompensation in the iteration domain* (see Figures 8) as an *ad hoc* fix to recover tracking performance. Pole placement methods typically change DC gains of systems, which are commonly recovered through precompensation, and this idea can be easily extended to ILC systems. One simple choice for the precompensator K is given by inverting the nominal steady state reference to output matrix given in (14),

$$K = (\bar{P}(I - (Q - L\bar{P}))^{-1}L)^{-1}, \quad (21)$$

which is verified experimentally: Figure 9 shows that precompensation results in approximately an order of magnitude improvement in tracking, i.e. an order of magnitude decrease in the norm of the error $r - y_k$. Moreover, the precompensated system maintains stability in the presence of perturbations, as can be seen in Figure 10. However, we emphasize that by virtue of its open loop (in the iteration domain) nature, the performance of a precompensated ILC scheme depends largely on the accuracy of the modeling information; in our case, the accuracy of (21).

8. Conclusion

In this paper, we scrutinized the stability and convergence properties of ILC systems subject to trial to trial uncertainty. We formulated the system to be controlled as a linear input-output map in an abstract Banach space setting to ensure the generality of our anal-

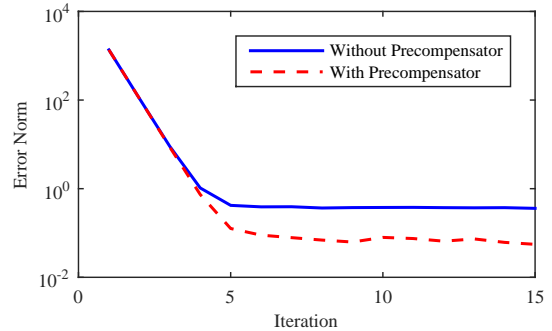


Figure 9: Experimental verification of precompensated norm optimal ILC, with weights $w_e = 10$, $w_u = 0.0025$, and $w_{\Delta u} = 1$. Precompensation leads to an order of magnitude decrease in the norm of the error $r - y_k$ as the iteration index $k \rightarrow \infty$.

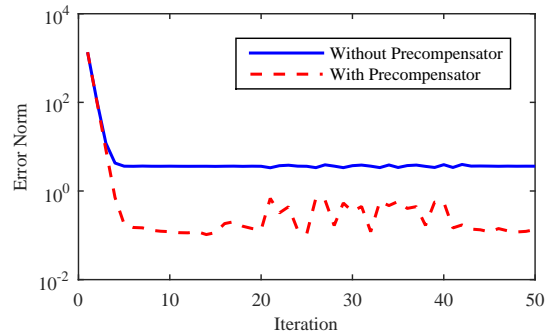


Figure 10: Experimental verification of precompensated norm optimal ILC subject to mass perturbation sequence \mathcal{S} with weights $w_e = 10$, $w_u = 0.0025$, and $w_{\Delta u} = 1$. Precompensation leads to more than an order of magnitude decrease in the norm of the error $r - y_k$ as the iteration index $k \rightarrow \infty$, and does not affect robust stability.

ysis, assuming bounded uncertainties in all process parameters; including the input-output operator, reference, disturbances and initial conditions. We showed that when a linear update law is designed to be robust over the uncertainty set \mathcal{P} , linear discrete time methods can be employed directly to show the system exhibits desirable properties such as asymptotic stability and boundedness. Moreover, we investigated how the design of the operators Q and L affects the convergence properties of iteration varying systems. We showed that an iteration varying system converges to 1) a bounded neighborhood of a nominal system if the uncertainties are bounded, and, 2) the nominal system itself if the uncertainties are convergent. Further we argued for employing an optimization based approach to ILC design to improve predictability in iteration varying systems. Our analysis was supported by simulation results, along with experimental verification on a linear motion control stage.

It turns out that robust ILC methods, which are well studied in the literature, can be applied directly to iteration varying systems. The results are strong in terms of their generality and the lack of limiting assumptions apart from linearity. A further direction to pursue is the study of optimal ILC strategies with structured (time invariant, higher order etc.) perturbations under discrete or continuous frameworks, with or without feedback. A disturbance rejection problem has been considered in Moore and Verwoerd (2008) via l_1 norm minimization, and an \mathcal{H}_∞ minimization problem for HOIM based plants, references, disturbances has recently been considered in Zhu et al. (2015). We expect the initial results of our paper, along with some of the work in Moore and Verwoerd (2008); Zhu et al. (2015) to pave the way for future research in iteration varying systems in ILC.

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Appendix A. On the Banach Space Framework

As stated in Section 2, the motivation behind the abstract Banach space framework is to preserve some generality: The operator \bar{P} represents the input-output relationship of a linear system, which can be described by an ordinary differential equation, a partial differential equation, or a difference equation, over a finite or infinite domain. This operator can be causal/noncausal and invariant/varying with respect to the underlying independent variables (e.g. time and/or space). In addition, the input-output spaces can be multidimensional. For example, the operator \bar{P} can be a stable causal $n \times n$ transfer matrix, in which case the input-output spaces can be defined as square integrable signals in the frequency domain (i.e. $U = Y = \mathcal{H}_2^n$) with the corresponding induced norm as the \mathcal{H}_∞ norm. Hence, by taking U and Y as complete normed spaces, and $\bar{P} : U \rightarrow Y$ as a bounded linear operator, we will be able to have a complete analysis valid for a broad class of problems, in a simplified fashion.

Care must be taken in the definitions of the operators, as boundedness depends on the specific choice of spaces. Two examples are given below.

Example 1. Consider the scalar differential equation $\dot{y}(t) = cy(t) + u(t)$ with the initial condition $y(0) = 0$, where $c \in \mathbb{R}$, and $U = Y$ is the space of continuous functions over the interval $[a, b]$ with the sup norm. The differential equation is a bounded operator for

- $a = 0, b = \infty$, if and only if $c < 0$,
- $a = -\infty, b = 0$, if and only if $c > 0$, and
- $a, b \in \mathbb{R}$, for all c .

Example 2. Consider the convolution operator represented by the transfer function $1/(1-s)$. This operator is unbounded (unstable) if the transfer function is the one sided Laplace transform (over the positive real line) of the kernel e^t , but is bounded (stable) if it is the bilateral Laplace transform of the kernel $e^t \mathbf{1}(-t)$, where $\mathbf{1}(\cdot)$ is the Heaviside step function. Here, the input-output spaces are \mathcal{L}_p for any $p \in [1, \infty]$.

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