Hybrid Systems with Delayed Jumps: Asymptotic Stability via Robustness and Lyapunov Conditions

Berk Altın, *Member, IEEE,* and Ricardo G. Sanfelice, *Senior Member, IEEE*

***Abstract*—Hybrid systems subject to delayed jumps form a class of dynamical systems with broad applications. This paper develops sufficient conditions for robust asymptotic stability of hybrid systems in the presence of delayed jumps. More precisely, given a delay-free hybrid system, we introduce a higher order delayed system parametrized by the length of delays. We show that when the delay parameter is set to zero, the higher order model captures the solutions of the delay-free system. Under mild conditions, it is shown that when the delay-free system has an asymptotically stable compact set, for small enough delays, solutions of the delayed system converge to a neighborhood of a set of interest related to the aforementioned compact set. Then, Lyapunov functions for the delay-free system are used to develop sufficient conditions for asymptotic stability in the presence of delays. Unlike prior work in the literature, the results pertaining to these notions of stability hold for systems possessing Zeno solutions, with time-varying delays. Importantly, the required conditions are expressed in finite-dimensional space, and depend primarily on the data of the delay-free system. The practical stability result is validated numerically through the hybrid system model of a controlled boost converter circuit with state-triggered switches and Zeno solutions. The higher order model and the derived Lyapunov conditions are utilized to obtain quantitative bounds on maximum allowable delays for switched systems and sampled-data control.**

I. Introduction

For hybrid systems in the framework of [1], where a hybrid system is described by a combination of constrained differential and difference inclusions, we study the effects of delays on stability properties. The modeling approach adopted in [1] encapsulates a diverse set of related frameworks such as hybrid automata, impulsive differential equations, and switched systems, and emphasizes robustness of asymptotic stability to external perturbations under standard regularity conditions. The primary objective of this paper is to extend these results by scrutinizing the effects of a class of delays, and develop sufficient conditions for asymptotic stability in the presence of said delays.

A. Motivation

The present work is motivated largely by cyber-physical systems with delay phenomena arising from communication constraints and computational limitations, and focuses on asymptotic stability properties of hybrid dynamical systems in the presence of delays on events, or *jumps*. As an example, consider a continuous-time control system with state $x_p \in \mathbb{R}^{n_p}$ and input $u \in \mathbb{R}^{n_c}$, governed by a vector field f. Then, given a state-feedback law κ and a sampling period $T_s > 0$, a sample-and-hold implementation of κ can be modeled as a hybrid system by treating u as a state variable and introducing the sampling timer τ_s. In particular, the state $x = \begin{bmatrix} x_p^\top & \tau_s^\top & u^\top \end{bmatrix}^\top$ of the sampled-data control system evolves according to the continuous dynamics

$$\dot{x} = F(x) := \begin{bmatrix} f(x_p, u) \\ 1 \\ 0 \end{bmatrix} \quad x \in C := \mathbb{R}^{n_p} \times [0, T_s] \times \mathbb{R}^{n_c}, \tag{1}$$

and the discrete dynamics

$$x^+ = G(x) := \begin{bmatrix} x_p \\ 0 \\ \kappa(x_p) \end{bmatrix} \quad x \in D := \mathbb{R}^{n_p} \times \{T_s\} \times \mathbb{R}^{n_c}, \tag{2}$$

describing the following principles. When $\tau_s \in [0, T_s]$, the timer variable counts up at a constant rate of one until it reaches T_s, at which point it resets to zero. In the intersample period when τ_s evolves continuously, or equivalently, *flows*, the plant state changes according to the differential equation $\dot{x}_p = f(x_p, u)$, while u stays constant. When the timer τ_s resets to zero, the input u is updated to $\kappa(x_p)$. A precise definition of solutions for this system is provided in Section II.

Under appropriate assumptions, robustness results in [1] help show that the system represented by (1) and (2) can not only tolerate disturbances, noise, and uncertainties on f, but also time-varying uncertainties on the sampling times. However, because of computational limitations, in practice, there is a strictly positive amount of time between the timer update event $\tau_s^+ = 0$ and control update event $u^+ = \kappa(x_p)$. Although the results in [1] do not directly address delays at jumps, and stability conditions for delayed systems in other frameworks rely on sophisticated infinite-dimensional certificates, due to the piecewise constant evolution of u, as in [2], the sampled-data control system with delays on the update of u can be described by a higher order hybrid system. We generalize this approach to analyze the effects of delayed jumps on stability of hybrid systems, and derive finite-dimensional Lyapunov stability conditions for full compensation of delays. In addition to sampled-data control, as demonstrated throughout the paper, the results we develop are related to a wide variety of applications such as biological systems, state observers, power conversion circuits, and mechanical systems with impacts.

B. Related Work and Contributions

The study of delay phenomena in the systems and control literature can be traced back to the early works of Razumikhin

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and Krasovskii in the late 1950s and early 1960s, respectively. A detailed review of the large body of literature on this subject is outside the scope of this paper, and readers are referred to the survey [3] and the recent books [4]–[6]. In the context of hybrid systems, there have been a number of works studying delays in various settings. In an extension of the abstract time-space framework of [7], early works pertaining to hybrid systems with delays have concentrated on certifying asymptotic stability via Razumikhin functions [8], [9]. Stability certificates for impulsive retarded functional differential equations that rely on Lyapunov functionals have been reported in [10]. For switched systems, a minimum dwell-time asymptotic stability condition is given in [11], under the assumption that each mode is linear, subject to fixed delays, and delay dependently or independently stable. Input-to-state and integral input-to-state stability properties of impulsive systems, and switched impulsive systems, have also been explored using Razumikhin functions [12], and dwell-time restrictions via Lyapunov-Krasovskii functionals [13], [14]. Finally, delay dependent and independent stability of linear reset systems is studied in [15], [16], through the use of passivity, linear matrix inequalities (LMIs), and the Kalman-Yakubovich-Popov lemma.

Unfortunately, the majority of the aforementioned works in the delayed hybrid systems literature fail to capture the generality associated with the hybrid inclusions formalism, and concentrate on specific hybrid models (e.g. impulsive systems or switched systems). Although the valuable work extending hybrid inclusions to the delayed case in a series of recent articles [17]–[21] present an opportunity to study the effects of delay in a more general sense, the sufficient conditions for robustness in the *hybrid systems with memory* framework imposes requirements on the system data with respect to the distance functions [22] on the underlying infinite-dimensional space, which can be hard to check. On the contrary, the conditions we impose in this paper for robustness against delays are commonly used to certify robustness with respect to a very general class of perturbations. In particular, with the observation that many (hybrid) controllers are designed and analyzed in a delay-free setting, the principal contribution of this paper is to present sufficient conditions that guarantee robust asymptotic stability in the presence of delayed jumps, but depend primarily on the delay-free model. More specifically, denoting by $\mathcal{H}$ the delay-free model, our contributions are summarized as follows:

(i) Section III presents a parametric higher order model $\mathcal{H}'_T$ capturing the behavior $\mathcal{H}$ when it experiences delays on jumps, where T denotes the length of delays. Equivalence between solutions of $\mathcal{H}$ and $\mathcal{H}'_0$ ($\mathcal{H}'_T$ for the case of $T = 0$) is then used to show that given an asymptotically stable set $\mathcal{A}$ for $\mathcal{H}$, there exists an asymptotically stable set $\mathcal{A}'$ for $\mathcal{H}'_0$ (Section IV). This fact is utilized in Section V to prove that under mild regularity conditions on $\mathcal{H}$, asymptotic stability of $\mathcal{A}'$ is practically robust; that is, given a compact set K and $\varepsilon > 0$, there exists $T > 0$ such that solutions of $\mathcal{H}'_T$ originating from K converge to an ε-neighborhood of $\mathcal{A}'$.

(ii) In Section VI, given minimum and maximum length of delays, $T_{\min}$ and $T_{\max}$, respectively, a nonparametric higher order model denoted $\widetilde{\mathcal{H}}$ is used to derive sufficient conditions for asymptotic stability of a set $\widetilde{\mathcal{A}}$, which is related to $\mathcal{A}$. The model depends on $T_{\min}$ and $T_{\max}$, but is not explicitly parametrized by these parameters. The sufficient conditions presented here exploit the availability of a Lyapunov function V that can be used to certify asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$. Later, these conditions and the nonparametric higher order model are utilized in Section VII to come up with constructive bounds on allowable delays for switched systems and sampled-data control systems.[1]

Different from the large body of literature on delay systems and related fields, the result discussed in (i) shows that for delay-free finite-dimensional hybrid systems, asymptotic stability is semiglobally practically robust with respect to the length of delays. The significance of this result, though semiglobal and practical,[2] is brought out by the fact that it holds for the case of time-varying delays, without any dwell-time restrictions. The relevance of the latter feature is highlighted through the hybrid controlled boost converter possessing Zeno solutions, presented in Section VII: as we shall see, any stabilizing hybrid feedback for this circuit necessarily produces Zeno solutions. In this regard, the importance of practical robustness cannot be overstated, as Zeno solutions cannot be realized in the presence of switching delays.

The class of delayed hybrid systems studied in this paper pertains to a large variety of applications. They model time-delays appearing naturally in feedback loops interconnected over networks [3], due to communication constraints, computational limitations, and actuator dynamics. Delay-free hybrid systems are also used to model dynamical systems with continuous changes at vastly different time-scales, with discrete dynamics modeling changes at the fast time-scale. For example, colliding rigid bodies are often modeled as hybrid systems, since the strictly positive duration of impacts is very short [26]. Existence of formal robustness guarantees against delayed jumps has important connotations in justifying this abstraction of a discrete approximation of the faster scale.

II. Preliminaries

Throughout the paper, we use $\mathbb{R}$ to represent real numbers, $\mathbb{R}_{\geq 0}$ its nonnegative subset, and $\mathbb{N}$ the set of nonnegative integers. Given a vector $x \in \mathbb{R}^n$ and a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |x - a|$ denotes the distance of x to $\mathcal{A}$, where $|.|$ is the 2-norm. We denote by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^n$, and by $\mathcal{A} + \delta\mathbb{B}$ the set of all $x \in \mathbb{R}^n$ such that $|x - a| \leq \delta$ for some $a \in \mathcal{A}$. $S_1 \subset S_2$ indicates that S_1 is a subset of S_2, not necessarily proper. The interior and closure of a set S are denoted $\operatorname{int} S$ and $\operatorname{cl} S$, respectively. The notation $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ indicates a set-valued mapping, in other words, for every $x \in \mathbb{R}^n$, $M(x)$ is a subset of $\mathbb{R}^m$.

[1]The preliminary work in [23] does not include these results. It only states the semiglobal practical stability result in (i) without proof, under more restrictive assumptions.

[2]Similar analysis has been shown to be useful in examining the effects of discretization [24] and actuator dynamics [25] on hybrid systems.

The domain of M, i.e., the set of all x such that $M(x)$ is nonempty, is denoted $\operatorname{dom} M$. The zero vector in $\mathbb{R}^n$ is denoted 0_n, or simply 0 when appropriate. Given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $(x, y) = \begin{bmatrix} x^\top & y^\top \end{bmatrix}^\top$.

In addition, we use the following comparison function definitions. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is strictly increasing, unbounded, and $\alpha(0) = 0$. Similarly, a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0} \beta(r, s) = 0$ for all $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s) = 0$ for all $r \in \mathbb{R}_{\geq 0}$.

A. Hybrid Inclusions and their Solutions

This paper considers hybrid systems in the framework of [1], where a hybrid system $\mathcal{H}$ is identified by the 4-tuple (C, F, D, G) (referred to as the data of $\mathcal{H}$), and described in the following form:

$$\mathcal{H} \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases} \tag{3}$$

Equivalently, we use the notation $\mathcal{H} := (C, F, D, G)$ to refer to the hybrid system in (3) and define its data.

The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called the *flow map*, and it describes the continuous evolution (*flows*) of the state $x \in \mathbb{R}^n$ on the *flow set* $C \subset \mathbb{R}^n$. Similarly, the set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ (the *jump map*) describes the discrete evolution (*jumps*) of x on the *jump set* $D \subset \mathbb{R}^n$. For the model to be meaningful, it is also assumed that $C \subset \operatorname{dom} F$ and $D \subset \operatorname{dom} G$.

Solutions of the hybrid system $\mathcal{H}$ are parametrized by the pair $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t is the ordinary time keeping track of the flows, and j is the jump index counting the number of jumps. The domain $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ of a solution x of $\mathcal{H}$ is a *hybrid time domain*, in the sense that for every $(T, J) \in \operatorname{dom} x$, there exists a nondecreasing sequence $\{t_j\}_{j=0}^{J+1}$ with $t_0 = 0$ such that

$$\operatorname{dom} x \cap ([0, T] \times \{0, 1, \ldots, J\}) = \cup_{j=0}^{J} ([t_j, t_{j+1}] \times \{j\}).$$

Given the sequence above, for any $j \in \{1, 2, \ldots, J\}$, t_j is the ordinary time of the j-th jump of x.

Definition 2.1: A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is said to be a solution of the hybrid system $\mathcal{H}$ if $\operatorname{dom} x$ is a hybrid time domain, $x(0, 0) \in \operatorname{cl}(C) \cup D$, and the following hold:

- For all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \operatorname{dom} x\}$ has a nonempty interior, the function $t \mapsto x(t, j)$ is locally absolutely continuous on I^j, $x(\operatorname{int} I^j, j) \subset C$, and
$$\dot{x}(t, j) \in F(x(t, j)) \text{ for almost all } t \in I^j.$$
- For all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,
$$x(t, j) \in D \text{ and } x(t, j + 1) \in G(x(t, j)).$$

A solution x of $\mathcal{H}$ is called complete if its domain is unbounded, and maximal if it cannot be extended to another solution. It is said to be Zeno if it is complete and $\sup_{(t,j)\in\operatorname{dom} x} t$ is finite, and eventually continuous if $J := \sup_{(t,j)\in\operatorname{dom} x} j$ is finite and the interval I^J defined above has nonzero length. The notation $\mathcal{S}_\mathcal{H}(S)$ refers to the set of all maximal solutions x of $\mathcal{H}$ originating from S; i.e. $x(0, 0) \in S$ for every $x \in \mathcal{S}_\mathcal{H}(S)$. The set of all maximal solutions of $\mathcal{H}$ is simply denoted $\mathcal{S}_\mathcal{H}$.

B. Basic Assumptions from Set-Valued Analysis

Well-posed hybrid systems [1, Definition 6.29] refer to a class of systems described as in (3) with numerous useful properties, chief among them robustness to perturbations on the data. One of the main objectives of this article is to show that in addition to the usual robustness property, under certain conditions, well-posed hybrid systems are robust with respect to delayed jumps.

Verifying well-posedness can be a difficult task. However, a set of mild regularity conditions on the data, called *hybrid basic conditions*, prove to be sufficient for this property [1, Th. 6.30], and in fact, for the aforementioned robustness property with respect to delays. Prior to stating the hybrid basic conditions, two concepts for set-valued mappings, *local boundedness* and *outer semicontinuity*, are introduced.

A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be locally bounded relative to a set $S \subset \mathbb{R}^n$ if for all $x \in \mathbb{R}^n$, there exists $\varepsilon > 0$ such that the set $M((x + \varepsilon\mathbb{B}) \cap S)$ is bounded. We omit a formal definition of outer semicontinuity (see [22, Definition 5.4] or [1, Definition 5.9]), but note that the mapping M is outer semicontinuous relative to S if and only if the graph of $M|_S$ (the restriction of M to S) is relatively closed in $S \times \mathbb{R}^m$ [1, Lemma 5.10]. That is, M is outer semicontinuous relative to S if and only if there exists a closed set K such that $\{(x, y) : x \in S, y \in M(x)\} = K \cap (S \times \mathbb{R}^m)$. In particular, M is locally bounded and outer semicontinuous relative to S if it is single valued and continuous on S.

Assumption 2.2 (Hybrid Basic Conditions): The following hold for the data of $\mathcal{H}$:

(A1) The sets C and D are closed.

(A2) The flow map F is locally bounded and outer semicontinuous relative to C, and $C \subset \operatorname{dom} F$. Furthermore, for every $x \in C$, the set $F(x)$ is convex.

(A3) The jump map G is locally bounded and outer semicontinuous relative to D, and $D \subset \operatorname{dom} G$.

The conditions outlined above combine what is typically assumed in continuous- and discrete-time systems. As can be seen throughout the paper, the behavior of many dynamical systems can be described in the form of (3) while satisfying the hybrid basic conditions. For the case of a flow map F that is single valued on C, Condition (A2) takes a simple form: if F is given by a function $f : C \to \mathbb{R}^n$ on C, (A2) holds if and only if f is continuous. The same comment applies to the jump map G and Condition (A3). For hybrid systems violating these conditions, the corresponding Krasovskii regularization [1, Definition 4.13] satisfies Assumption 2.2, provided F and G satisfy the local boundedness requirements in (A2) and (A3), respectively [1, Example 6.6]. See, for example, the boost converter circuit in Section VII.

III. Modeling of Delayed Jumps in Hybrid Systems

This section details the construction of a hybrid system modeling delayed jumps for $\mathcal{H}$. The constructed system, denoted $\mathcal{H}'_T$, depends on the parameter $T \geq 0$ meant to capture the maximum length of delays. When $T = 0$, this construction can be viewed as a redundant, higher order representation of the delay-free system $\mathcal{H}$.

To construct the high-dimensional model subject to delays on jumps, we introduce the decomposition of the state as $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ for some nonzero n_2, where x_2 represents the state components subject to delayed jumps.[3] This formulation is motivated by cyber-physical systems, wherein measurements and/or control inputs might be subject to computational delays, while the physical plant state evolves delay-free.

Example 3.1 (Sampled-Data Control): Consider the hybrid model of the sampled-data control system in (1)-(2). Since the input u is subject to delays, while the plant state x_p and the sampling timer τ_s evolve without any delays, the closed-loop state can be partitioned so that $x_1 = (x_p, \tau_s)$ and $x_2 = u$. Note also that the data in (1)-(2) satisfies Assumption 2.2 when f and κ are continuous, as C and D are closed, and F and G are single valued.

Certain biological networks can also be modeled as hybrid inclusions to certify robustness with respect to delays. These delays can arise from communication latency in between agents, or nonzero reaction time to stimuli, as discussed next.

Example 3.2 (Flashing Fireflies): Let $g : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued mapping such that $g(y) = y$ if $y < 1$, $g(y) = 0$ if $y > 1$, and $g(1) = \{0, 1\}$. Given $\varepsilon > 0$, let

$$\begin{aligned} C &= [0,1] \times [0,1], \\ F(x) &= (1,1) \quad \forall x \in \mathbb{R}^2, \\ D &= \{(x_1,x_2) \in [0,1] \times [0,1] : \max\{x_1,x_2\} = 1\}, \\ G(x) &= g((1+\varepsilon)x_1) \times g((1+\varepsilon)x_2) \quad \forall x \in \mathbb{R}^2. \end{aligned}$$

The model described by the data above is derived from [1, Example 4.15], where x_1 and x_2 are the internal clock states of two flashing fireflies, with jumps corresponding to flashing events. During flows, the clock states increase with a steady rate of one. When a firefly reaches the flashing threshold of one, it resets its clock to zero, and the other reacts by adjusting its clock according to g. In terms of the state partioning discussed, we imagine a scenario where the firefly with state x_2 experiences a nonzero reaction time in response to flashing stimuli, while the other firefly can flash instantaneously.

Given a solution x of $\mathcal{H}$, at times, we denote by x_1 and x_2 the same partition of x, in the sense that $x = (x_1, x_2)$ for some $x_1 : \operatorname{dom} x \to \mathbb{R}^{n_1}$ and $x_2 : \operatorname{dom} x \to \mathbb{R}^{n_2}$.

A. Higher Order Modeling of Jump Delays

Following the partitioning of the state into undelayed and delayed components, let $\widehat{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$ be a set-valued mapping such that $\widehat{G}(x) = P(G(x) \times \{x_2\})$ for every $x = (x_1, x_2) \in \mathbb{R}^n$, where

$$P(y_1, y_2, x_2) = (y_1, x_2, y_2) \quad \forall (y_1, y_2, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}. \tag{4}$$

In other words, given $(y_1, y_2, x_2) \in G(x_1, x_2) \times \{x_2\}$, the mapping P switches the coordinates corresponding to the delayed state x_2 and its post-jump value y_2. The following lemma is immediate.

Lemma 3.3: The mapping $\widehat{G}$ is locally bounded and outer semicontinuous relative to D if and only if the jump map G is locally bounded and outer semicontinuous relative to D.

Next, given a class-$\mathcal{K}_\infty$ function α and a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, for every $T \geq 0$, let

$$\begin{aligned} C_T &:= \{x \in \mathbb{R}^n : (x + \alpha(T)\rho(x)\mathbb{B}) \cap C \neq \varnothing\}, \\ F_T(x) &:= F((x + \alpha(T)\rho(x)\mathbb{B}) \cap C) + \alpha(T)\rho(x)\mathbb{B} \quad \forall x \in \mathbb{R}^n. \end{aligned} \tag{5}$$

Using the constructions in (5), the delayed hybrid system $\mathcal{H}'_T := (C'_T, F'_T, D'_T, G'_T)$ with state $z = (x', \mu, \tau)$, where $x' = (x'_1, x'_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $\mu \in \mathbb{R}^{n_2}$, and $\tau \in \mathbb{R}$, is described by the data below:

$$C'_T := (C_T \times \{0_{n_2}\} \times \{-1\}) \cup (C_T \times \mathbb{R}^{n_2} \times [0, T]), \tag{6}$$

$$F'_T(z) := F_T(x') \times \{0_{n_2}\} \times \{-\min\{\tau + 1, 1\}\} \quad \forall z \in C'_T, \tag{7}$$

$$D'_T := (D \times \{0_{n_2}\} \times \{-1\}) \cup (\mathbb{R}^n \times \mathbb{R}^{n_2} \times \{0\}), \tag{8}$$

and for every $z \in D'_T$,

$$G'_T(z) := \begin{cases} \widehat{G}(x') \times [0, T] & \text{if } z \in D \times \{0_{n_2}\} \times \{-1\} \\ (x'_1, \mu, 0_{n_2}, -1) & \text{if } z \in \mathbb{R}^n \times \mathbb{R}^{n_2} \times \{0\}. \end{cases} \tag{9}$$

The state of $\mathcal{H}'_T$ has three components. The first one, x', corresponds to the state x of the original delay-free system $\mathcal{H}$.[4] The component τ is a timer state regulating delays; $\tau \geq 0$ implies that the delay is due to expire in τ units of ordinary time, while $\tau = -1$ indicates that delays are inactive. The memory state μ records the post-jump value of the delayed state x'_2; this is described in (9) by the mapping $\widehat{G}$. More specifically, the jump map G'_T implements the following mechanism:

- When $x' = (x'_1, x'_2)$ belongs to the jump set D and there is no active delay, a jump records the delay-free post-jump value of the delayed state x'_2 in the memory state μ, and activates the delay timer.
- When the delay timer expires, i.e., $\tau = 0$, a jump deactivates the delay dynamics and updates the value of x'_2 to μ.

The construction in (5) is derived from the definition of a generic perturbed hybrid system [1, Definition 6.27] with perturbation function ρ. Given any ρ, the set C_T increases in T, in the sense that $C_{T_1} \subset C_{T_2}$ if $T_1 \leq T_2$. The same property holds for $F_T(x)$, for every $x \in \mathbb{R}^n$. This implies that every solution z of $\mathcal{H}'_{T_1}$ is a solution of $\mathcal{H}'_{T_2}$. The role of this construction is to introduce generality to the hybrid system $\mathcal{H}'_T$ so that a realistic model of delays can be obtained under different circumstances. This is illustrated by the following examples.

[3] While it is implicitly assumed that $n_1 \geq 0$, the results of this paper hold with appropriate modifications.

[4] We use the notation x' to differentiate it with x, since at times, we compare solutions of $\mathcal{H}'_T$ to solutions of $\mathcal{H}$.

Example 3.4 (Sampled-Data Control with Delays): Consider the data of sampled-data control system of Example 3.1. Let $\rho(x) = 0$ for all $x \in \mathbb{R}^{n_p+n_c+1}$. Then,

$$C'_T = (C \times \{0_{n_c}\} \times \{-1\}) \cup (C \times \mathbb{R}^{n_c} \times [0, T]),$$

and for every $z \in \mathbb{R}^{n_p+n_c+1} \times \mathbb{R}^{n_c} \times \mathbb{R}$,

$$F'_T(z) = F(x) \times \{0_{n_c}\} \times (-\min\{\tau + 1, 1\}).$$

Immediately, it is clear that the resulting $\mathcal{H}'_T$ is a sufficiently realistic representation of the sampled-data control system when $T < T_s$. In particular, one can observe that after every sampling event, since τ_s is reset to zero, the solution can flow in C until τ reaches zero, as f is defined globally and $T < T_s$.

Example 3.5 (Flashing Fireflies with Delays): Consider the data of the flashing fireflies network of Example 3.2. When the delay parameter T is nonzero, there exist solutions of $\mathcal{H}'_T$ that jump from $(1, 1, 0, -1)$ to a point $(0, 1, 0, \tau)$ with nonzero τ, due to (9). If ρ is the zero function, $C_T = [0, 1] \times [0, 1]$, and $F_T(x) = (1, 1)$ for all $x \in \mathbb{R}^2$. In this case, the aforementioned solutions cannot flow from $(0, 1, 0, \tau)$, and therefore must terminate.

To allow such solutions to flow, ρ can be chosen as a nonzero constant, in which case $C_T = C + \alpha(T)\rho\mathbb{B}$, and for every $x \in C_T$, $F_T(x) = \{(1, 1)\} + \alpha(T)\rho\mathbb{B}$. For instance, if ρ is constant and $\alpha(T) = \sqrt{T}/\rho$ for all $T \geq 0$, the distance of any point on the boundary of C to the boundary of C_T is $\sqrt{T}$. Moreover, the magnitude of the velocity $\dot{x} \in F(x)$ is upper bounded by $\sqrt{2} + \sqrt{T}$. Then, there exists $T > 0$ such that $T(\sqrt{2}+\sqrt{T}) \leq \sqrt{T}$. Therefore, given such T, every maximal solution $z = (x', \mu, \tau)$ of $\mathcal{H}'_T$ satisfying $\tau(0, 0) \in [0, T]$ and $x'(0, 0) \in C$ can flow for $\tau(0, 0)$ units of ordinary time.

Remark 3.6: As shown in Example 3.5, with careful selection of ρ and α, it can be guaranteed that solutions of $\mathcal{H}$ originating from $\operatorname{cl} C$ can flow for T units of ordinary time. Hence, an additional assumption on G of the form

$$(x_1, x_2) \in D \text{ and } (y_1, y_2) \in G(x) \implies (y_1, x_2) \in \operatorname{cl} C$$

can be imposed for flows to be always possible during delays. While the jump maps in Examples 3.1 and 3.2 both satisfy this property, we do not explicitly assume it as it does not affect the technical results. Similarly, for tighter constraints on the memory state μ, the set $\mathbb{R}^{n_2}$ in (6) and (8) can be replaced with a closed set $\mathcal{M}$ such that

$$(x_1, x_2) \in D \text{ and } (y_1, y_2) \in G(x) \cap (\operatorname{cl}(C) \cup D) \implies y_2 \in \mathcal{M}.$$

As before, similar to the notation we follow for solutions of $\mathcal{H}$, given a solution z of $\mathcal{H}'$, x', μ, and τ denote the components of z corresponding to the state of $\mathcal{H}$, the memory state, and timer state, respectively. From hereinafter, the hybrid system $\mathcal{H}' := (C', F', D', G')$ denotes $\mathcal{H}'_T$ for the case $T = 0$, where $C' := C'_0$, $F' := F'_0$, $D' := D'_0$, and $G' := G'_0$.

It is worth noting that the augmented jump map introduced in (9) encodes a sequential execution of jumps. That is, it does not allow solutions to jump due to the state component x' reaching D during an active delay. The modeling decision here is justified for small enough T when the ordinary time interval between jumps is uniformly lower bounded by a positive constant over the set of maximal solutions of $\mathcal{H}$. This fact is obvious for the sampled-data control system discussed in Example 3.1 since jumps are separated by T_s units of time, but as shown in Appendix B, it also holds semiglobally for the generic hybrid system $\mathcal{H}$ under Assumption 2.2, provided $\mathcal{H}$ has a pre-asymptotically stable (defined in Section IV) compact set, and $G(D) \cap D$ is empty (i.e., consecutive jumps are not allowed). If the set $G(D) \cap D$ is nonempty, the model can be justified if $\mathcal{H}$ arises from the interconnection of two hybrid systems, as is the case for the flashing fireflies network.

The following proposition shows that $\mathcal{H}'$ is a higher order representation of $\mathcal{H}$, thereby justifying our study of $\mathcal{H}'$ (and by extension, its perturbation $\mathcal{H}'_T$), in order to assess the robustness of the original hybrid system $\mathcal{H}$ against delays. It states that for every solution x of $\mathcal{H}$, there exists a solution $z = (x', \mu, \tau)$ of $\mathcal{H}'$ satisfying the following:

- It flows when x flows, and jumps twice each time x jumps.
- During flows at ordinary time t, the x' component of z is identical to x, while the memory state μ and delay timer τ stay constant at 0 and -1, respectively.

Proposition 3.7: Given a solution $x = (x_1, x_2)$ of $\mathcal{H}$, let $\{t_j\}_{j=1}^{J}$ be the sequence[5] of jump times of x, i.e.,

$$(t_j, j), (t_j, j-1) \in \operatorname{dom} x \quad \forall j \in \{1, 2, \ldots, J\} \cap \mathbb{N},$$
$$J := \sup\{j : \exists t, (t, j) \in \operatorname{dom} x\}.$$

Consider the function $z : \operatorname{dom} z \to \mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R}$, where

$$\operatorname{dom} z := \{(t, j) : (t, j/2) \in \operatorname{dom} x\} \cup \left(\cup_{j=1}^{J}\{(t_j, 2j-1)\}\right),$$

and for every $(t, j) \in \operatorname{dom} z$, $z(t, j) := (x(t, j/2), 0_{n_2}, -1)$ if j is even, and

$$z(t, j) := (x_1(t, (j+1)/2), x_2(t, (j-1)/2)$$
$$, x_2(t, (j+1)/2), 0)$$

if j is odd. Then, z is a solution to $\mathcal{H}'$. Moreover, z is maximal if x is maximal.

The proof of Proposition 3.7 can be found in Appendix A.

B. Key Properties of the Higher Order System

Having shown $\mathcal{H}'$ as a higher order representation of $\mathcal{H}$, we establish two properties of the delayed system $\mathcal{H}'_T$ that are used later for stability and robustness analysis. We begin with an analog of Proposition 3.7, which shows that for every solution of $\mathcal{H}'$ with delay timer component originating from -1, there exists a corresponding solution of $\mathcal{H}$. This result plays a key role in establishing pre-asympotic stability of an appropriately constructed set for $\mathcal{H}'$. The proof is similar to that of Proposition 3.7 and is omitted for brevity.

Lemma 3.8: Let $z = (x', \mu, \tau)$ be a maximal solution of $\mathcal{H}'$ satisfying $\tau(0, 0) = -1$, where $x' = (x'_1, x'_2)$. Then, for every $(t, j) \in \operatorname{dom} z$,

$$(\mu(t, j), \tau(t, j)) = \begin{cases} (0_{n_2}, -1) & \text{if } j \text{ is even} \\ (x'_2(t, j+1), 0) & \text{if } j \text{ is odd.} \end{cases}$$

[5]The sequence $\{t_j\}_{j=1}^{J}$ and the indexed union defining $\operatorname{dom} z$ are to be interpreted as empty if $J = 0$.

Moreover, the function $x : \operatorname{dom} x \to \mathbb{R}^n$*, where*

$$\operatorname{dom} x := \{(t, j) : (t, 2j) \in \operatorname{dom} z\}$$

and $x(t, j) := x'(t, 2j)$ *for all* $(t, j) \in \operatorname{dom} x$*, is a maximal solution of* $\mathcal{H}$*.*

Further equivalence between $\mathcal{H}$ and $\mathcal{H}'$ can be found in their regularity. For instance, it is obvious by inspection that C' is closed if and only if C is closed. The following lemma shows a similar property.

Lemma 3.9: Given any $T \geq 0$*, the augmented hybrid system* $\mathcal{H}'_T$ *satisfies the hybrid basic conditions if Assumption 2.2 holds, and the set* $F((x + \alpha_F(T)\rho(x)\mathbb{B}) \cap C)$ *is convex for all* $x \in C_T$*.*

Proof: That the sets C'_T and D'_T are closed follow from closedness of the sets C_T and D by Condition (A1). Indeed, if C is closed, C_T is precisely the set of all $x \in \mathbb{R}^n$ such that $|x|_C \leq \alpha(T)\rho(x)$, and continuity of ρ and $|.|_C$ implies that for every sequence $\{x_i\}_{i=0}^{\infty} \in C_T$ converging to a point x, $|x|_C \leq \alpha(T)\rho(x)$, i.e., $x \in C_T$. In addition, the flow map F'_T is locally bounded and outer semicontinuous relative to C'_T, as F_T is locally bounded and outer semicontinuous relative to C'_T, by virtue of Condition (A2) (see [1, Proposition 6.28]). Furthermore, $F'_T(z)$ is convex for all $z \in C'_T$ as $F_T(x)$ is convex for all $x \in C_T$. Finally, local boundedness and outer semicontinuity of G'_T is obvious since $\widehat{G}$ is locally bounded and outer semicontinuous by Lemma 3.3. ■

Corollary 3.10: The augmented delay-free hybrid system $\mathcal{H}'$ *satisfies the hybrid basic conditions if and only if Assumption 2.2 holds.*

If f and κ are continuous, the delayed sampled-data control system in Example 3.4 satisfies the hybrid basic conditions for $T = 0$. For the case of $T > 0$, it satisfies the hybrid basic conditions if f and κ are linear. More generally, given any ρ and α, $\mathcal{H}'_T$ satisfies the hybrid basic conditions if κ is continuous and the term $F((x + \alpha(T)\rho(x)\mathbb{B}) \cap C)$ in (5) is replaced with its convex closure. Given any ρ and α, the delayed flashing fireflies network discussed in Example 3.5 satisfies the hybrid basic conditions, as the flow map F is constant and the mapping g is outer semicontinuous.

IV. Stability of the Higher Order Delay-Free System

We now show that pre-asymptotic stability of a given closed set $\mathcal{A}$ for the hybrid system $\mathcal{H}$, defined in the usual δ-ε way, is preserved under the state augmentation that led to the delay-free system $\mathcal{H}'$, in an appropriate sense. Note that due to the existence of state variables that may not have an equilibrium, point stability is a restrictive notion for hybrid systems. For example, for the sampled-data control system in (1)-(2), it is impossible to stabilize any single point as the timer variable evolves in a periodic fashion. However, the controller κ can be designed to stabilize[6] the set $\mathcal{A} = \{0_{n_p}\} \times [0, T_s] \times \{\kappa(0_{n_p})\}$.

Definition 4.1: A closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be stable for the hybrid system $\mathcal{H}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every solution x of $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} x$. It is said to be pre-asymptotically stable for $\mathcal{H}$ if it is stable for $\mathcal{H}$ and there exists $\sigma > 0$ such that every complete solution x of $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq \sigma$ satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$.

In Definition 4.1, the prefix "pre" is used to indicate that maximal solutions of $\mathcal{H}$ need not be complete. It is dropped when every maximal solution is complete.

Assumption 4.2: The set $\mathcal{A}$ is a pre-asymptotically stable for the hybrid system $\mathcal{H}$.

Given the pre-asymptotically stable closed set $\mathcal{A}$, the *basin of pre-attraction* of $\mathcal{A}$, denoted $\mathcal{B}^p_{\mathcal{A}}$, is the set of all $x_0 \in \mathbb{R}^n$ such that for every solution x of $\mathcal{H}$ originating from x_0, the function $(t, j) \mapsto |x(t,j)|_{\mathcal{A}}$ is bounded on $\operatorname{dom} x$, and if x is complete, then $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$. By definition, the basin of pre-attraction includes all points outside of $\operatorname{cl}(C) \cup D$. If $\mathcal{B}^p_{\mathcal{A}} = \mathbb{R}^n$, then $\mathcal{A}$ is said to be globally pre-asymptotically stable (for $\mathcal{H}$). When $\mathcal{A}$ is compact, the stability notions in Definition 4.1 and the definition of the basin of pre-attraction agree with [1, Definition 7.1] and [1, Definition 7.3], respectively.

In preparation for the ensuing analysis, it is necessary to construct a set $\mathcal{A}'$ embedding $\mathcal{A}$ into $\mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R}$ in a suitable manner so that pre-asymptotic stability can extend to the delay-free system $\mathcal{H}'$. Recalling (4), we let

$$\mathcal{A}' := (\mathcal{A} \times \{0_{n_2}\} \times \{-1\}) \cup (P(\mathcal{A} \times \operatorname{cl}(\mathcal{A}_2)) \times \{0\}), \quad (10)$$

where $\mathcal{A}_2$ is the projection of $\mathcal{A}$ onto $\mathbb{R}^{n_2}$. Note that $\mathcal{A}'$ is closed as $\mathcal{A}$ is closed. If $\mathcal{A}$ is compact, then so is $\mathcal{A}'$, since, in this case, the set $\operatorname{cl}(\mathcal{A}_2) = \mathcal{A}_2$ would be compact.

Intuitively, it is easy to see that $\mathcal{A}'$ should be pre-asymptotically stable for $\mathcal{H}'$ when Assumption 4.2 holds: solutions of $\mathcal{H}'$ originating from $\mathcal{A} \times \{0_{n_2}\} \times \{-1\}$ should flow in $\mathcal{A} \times \{0_{n_2}\} \times \{-1\}$ and jump onto $P(\mathcal{A} \times \operatorname{cl}(\mathcal{A}_2)) \times \{0\}$, while solutions originating from $P(\mathcal{A} \times \operatorname{cl}(\mathcal{A}_2)) \times \{0\}$ should jump onto $\mathcal{A} \times \{0_{n_2}\} \times \{-1\}$. As a matter of fact, pre-asymptotic stability of $\mathcal{A}'$ for $\mathcal{H}'$ is equivalent to that of $\mathcal{A}$ for $\mathcal{H}$.

Proposition 4.3: The set $\mathcal{A}'$ *is pre-asymptotically stable for the hybrid system* $\mathcal{H}'$ *if and only if Assumption 4.2 holds.*

Although a Lyapunov function-based stability analysis seems possible, the proof of this proposition, presented in Appendix A, does not utilize Lyapunov functions. There are several reasons for this approach, foremost of which is that pre-asymptotic stability of $\mathcal{A}$, as stated in Assumption 4.2, does not depend on the existence of a Lyapunov function, and converse Lyapunov theorems for hybrid systems [1, Th. 7.31, Corollary 7.32] assume further properties that are not required here. Moreover, it is not obvious how a Lyapunov function for $\mathcal{H}$ can be utilized to construct a Lyapunov function for the higher order system $\mathcal{H}'$ (and vice versa), due to the potential mismatch in the dimensions of the state components x' and μ. Instead, the proof relies on a general stability result targeted towards a class of hybrid systems that are stable when certain events are excluded—see Theorem C.1 in Appendix C and the discussion therein.

[6] See [1, Example 3.21] for the stability analysis of a sampled-data controller for a linear system using LMIs.

Remark 4.4: It is straightforward to check that the basins of pre-attraction $\mathcal{B}^p_{\mathcal{A}}$ and $\mathcal{B}^p_{\mathcal{A}'}$ satisfy

$$\begin{aligned}&(\mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R}) \setminus \mathcal{B}^p_{\mathcal{A}'} \\ &= ((\mathbb{R}^n \setminus \mathcal{B}^p_{\mathcal{A}}) \times \{0_{n_2}\} \times \{-1\}) \cup (P(\mathbb{R}^n \setminus \mathcal{B}^p_{\mathcal{A}}) \times \{0\}),\end{aligned}$$

which shows that global pre-asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$ is equivalent to global pre-asymptotic stability of $\mathcal{A}'$ for $\mathcal{H}'$. An alternative way of showing this fact is to modify the proof of Proposition 4.3, using Corollary C.2.

When $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ for some $\mathcal{A}_1 \subset \mathbb{R}^{n_1}$, the set $\mathcal{A}'$ in (10) does not differ much from $\mathcal{A}$. For example, in the case of the sampled-data controller, if $\mathcal{A} = \{0_{n_p}\} \times [0, T_s] \times \{\kappa(0_{n_p})\}$, asymptotic stability of $\mathcal{A}'$ for $\mathcal{H}'$ implies that the plant state should converge to the origin for the higher order model. This is not always the case, and the additional jumps embedded in the higher order model can affect the set of interest significantly. The flashing fireflies network in Example 3.2 has the set $\mathcal{A} = \{(x_1, x_2) \in C : x_1 = x_2\}$ asymptotically stable, but for the higher order representation, the clock states can only be guaranteed to converge to $\mathcal{A} \cup \{(0, 1)\}$.

V. Robustness against Delays in the Semiglobal Practical Sense

The goal of this section is to show that pre-asymptotic stability of the delay-free augmented hybrid system $\mathcal{H}'$ is robust, with respect to the perturbations described via the family of delayed hybrid systems $\mathcal{H}'_T$. This notion of robustness is related to semiglobal practical robust $\mathcal{KL}$ pre-asymptotic stability [1, Definition 7.18]. It guarantees that for any compact subset of the basin of pre-attraction and any $\varepsilon > 0$, there exists a maximum length of delay $T > 0$ for which pre-asymptotic stability is practically preserved. To establish this result, it is necessary to note that when $\mathcal{H}'$ is well-posed and the set $\mathcal{A}'$ is compact, Proposition 4.3 can equivalently be stated in terms of uniform bounds given by class-$\mathcal{KL}$ functions. In stating the results of this section, we rely on *proper indicators*.

Definition 5.1: Given an open set $\mathcal{U} \subset \mathbb{R}^n$, a continuous function $\omega : \mathcal{U} \to \mathbb{R}_{\geq 0}$ is a proper indicator of a compact set $\mathcal{A} \subset \mathcal{U}$ on $\mathcal{U}$ if the following hold:

- $\omega(x) = 0$ if and only if $x \in \mathcal{A}$.
- Given any sequence $\{x_i\}_{i=0}^{\infty} \in \mathcal{U}$, $\lim_{i\to\infty} \omega(x_i) = \infty$ if $\lim_{i\to\infty} |x_i| = \infty$ or $\lim_{i\to\infty} |x_i|_{\mathbb{R}^n\setminus\mathcal{U}} = 0$.

In plain words, ω is a proper indicator of $\mathcal{A}$ if it is positive definite with respect to $\mathcal{A}$, and tends to infinity as its argument tends to infinity or the boundary of $\mathcal{U}$. When $\mathcal{A}$ is nonempty, a proper indicator of $\mathcal{A}$ on $\mathcal{U}$ is the function $\omega(x) := |x|_{\mathcal{A}}$ if $\mathcal{U} = \mathbb{R}^n$, and $\omega(x) := |x|_{\mathcal{A}} / (|x|_{\mathbb{R}^n\setminus\mathcal{U}})$ otherwise.

Proposition 5.2: Under Assumptions 2.2 and 4.2, if the set $\mathcal{A}$ is compact, the basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}'}$ of the set $\mathcal{A}'$ in (10) *is open. Moreover, for every proper indicator ω of $\mathcal{A}'$ on $\mathcal{B}^p_{\mathcal{A}'}$, there exists a class-$\mathcal{KL}$ function β such that every solution z of $\mathcal{H}'$ originating from $\mathcal{B}^p_{\mathcal{A}'}$ satisfies*

$$\omega(z(t,j)) \leq \beta(\omega(z(0,0)), t+j) \quad \forall (t,j) \in \operatorname{dom} z. \quad (11)$$

Proof: The set $\mathcal{A}'$ is compact when $\mathcal{A}$ is compact, and pre-asymptotically stable for $\mathcal{H}'$ by Proposition 4.3 when Assumption 2.2 holds. Furthermore, $\mathcal{H}'$ is satisfies the hybrid basic conditions by Corollary 3.10, and is therefore (nominally) well-posed. Hence, [1, Th. 7.12] applies, yielding the desired result. ■

Using Proposition 5.2, our first main result is given next. Note that openness of the basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}'}$ is crucial in making sense of this result, as it implies the existence of a neighborhood of the set $\mathcal{A}'$ contained in $\mathcal{B}^p_{\mathcal{A}'}$.

Theorem 5.3: Suppose that the conditions of Proposition 5.2 hold. Consider the basin of pre-attraction $\mathcal{B}^p_{\mathcal{A}'}$ of $\mathcal{A}'$, along with any proper indicator ω of $\mathcal{A}'$ on $\mathcal{B}^p_{\mathcal{A}'}$ and any class-$\mathcal{KL}$ function β satisfying (11) *for every solution z of $\mathcal{H}'$ originating from $\mathcal{B}^p_{\mathcal{A}'}$. Then, for every compact set $K \subset \mathcal{B}^p_{\mathcal{A}'}$ and every $\varepsilon > 0$, there exists $T > 0$ such that every solution z of $\mathcal{H}'_T$ originating from K satisfies*

$$\omega(z(t,j)) \leq \beta(\omega(z(0,0)), t+j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom} z. \quad (12)$$

Proof: As shown in the proof of Proposition 5.2, the augmented hybrid system $\mathcal{H}'$ is well-posed. Invoking [1, Lemma 7.20], $\mathcal{A}'$ is semiglobally practically robustly $\mathcal{KL}$ pre-asymptotically stable on $\mathcal{B}^p_{\mathcal{A}'}$. This implies that for every compact set $K \subset \mathcal{B}^p_{\mathcal{A}'}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that every solution z of $\mathcal{H}'_{\tilde{\rho}}$ with $z(0,0) \in K$ satisfies

$$\omega(z(t,j)) \leq \beta(\omega(z(0,0)), t+j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom} z,$$

where $\tilde{\rho} : \mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ is the function satisfying $\tilde{\rho}(z) := \delta\rho(x')$ for all $z = (x'\mu, \tau) \in \mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R}$ and $\mathcal{H}'_{\tilde{\rho}}$ is the $\tilde{\rho}$-perturbation (see [1, Definition 6.27]) of $\mathcal{H}'$. Finally, it is easy to verify that the inclusion $\mathcal{S}_{\mathcal{H}'_T} \subset \mathcal{S}_{\mathcal{H}'_{\tilde{\rho}}}$ holds for some $T > 0$, completing the proof. ■

We remind the reader that since the solution set of $\mathcal{H}'_T$ grows with T, Theorem 5.3 indicates a positive upper bound on the length of *time-varying* delays that the system can tolerate for solutions originating from K to converge to the ε-neighborhood of $\mathcal{A}'$, derived from (12). It is also worth pointing out that the hybrid system $\mathcal{H}'_T$ described in (6)-(9) is closely related to the notion of temporal regularization [27], and as a result of Theorem 5.3, hybrid systems with Zeno solutions satisfying Assumption 2.2 can be temporally regularized in practice by the introduction of time delays, while maintaining practical stability.

Remark 5.4: The semiglobal practical robustness property in Theorem 5.3 extends to more general perturbations of $\mathcal{H}'$, as can be seen in its proof. Moreover, if $\mathcal{H}'_T$ has a pre-asymptotically compact set for a given $T \geq 0$ and the conditions of Lemma 3.9 are satisfied, then pre-asymptotic stability of this set is robust, in the sense of Theorem 5.3. For example, if the vector field f in (1) and the control law κ in (2) are linear, and given $\mathcal{A} = \{0_{n_p}\} \times [0, T_s] \times \{\kappa(0_{n_p})\}$ and $T \geq 0$, the set

$$(\mathcal{A} \times \{0_{n_c}\} \times \{-1\}) \cup (P(\mathcal{A} \times \operatorname{cl}(\mathcal{A}_2)) \times [0, T]),$$

is pre-asymptotically stable for the hybrid system $\mathcal{H}'_T$ in Example 3.4, then the sample-and-hold controller with delays is robust with respect to small measurement noise.

In Section VII-A, Theorem 5.3 is applied to a closed-loop boost converter wherein the hybrid controller is subject to switching delays. This system is of particular interest since it has Zeno solutions, and it is impossible to achieve stability of the target set in the presence of delays.

VI. Asymptotic Stability in the Presence of Delays

Section V showed that pre-asymptotic stability of the set $\mathcal{A}$ for $\mathcal{H}$ leads to semiglobal practical stability in the presence of delays (in the sense of Theorem 5.3) when $\mathcal{A}$ is compact and $\mathcal{H}$ satisfies the hybrid basic conditions. The objective of this section is to provide sufficient conditions for pre-asymptotic stability with delays. While existing Lyapunov results [1, Ch. 3] can directly be applied to any higher order hybrid system representing the effects of delayed jumps on $\mathcal{H}$, following the approach that led to Theorem 5.3, we provide conditions that primarily depend on the data of $\mathcal{H}$ itself. In deriving the results of this section, we use a slightly different construction than in Sections III, IV, and V.

A. Construction of the Set to be Stabilized and the Higher Order Model

To formulate the model used in this section, unlike Sections III, IV, and V, we assume not just an upper bound, but also a lower bound on the length of delays; i.e., we assume that there exist $T_{\min} \geq 0$ and $T_{\max} \geq T_{\min}$ such that each jump induces a delay between $T_{\min}$ and $T_{\max}$. We do this to provide some generality, since, simply selecting $T_{\min} = 0$ and $T_{\max} = T$ recovers the previous case. As we do not explicitly impose any stability properties on $\mathcal{H}$, an added benefit of this approach is to check whether the introduction of delayed jumps can have a stabilizing effect on $\mathcal{H}$, a phenomenon previously observed in continuous time [3]. Consequently, we study pre-asymptotic stability of the set

$$\widetilde{\mathcal{A}} := (\mathcal{A}\times\{0_{n_2}\}\times\{-1\})\cup(P(\mathcal{A}\times\mathrm{cl}(\mathcal{A}_2))\times[0,T_{\max}]), \quad (13)$$

which is equal to the set $\mathcal{A}'$ in (10) when $T_{\max} = 0$. Pre-asymptotic stability of this set will be studied for the higher order system $\widetilde{\mathcal{H}} := (\widetilde{C}, \widetilde{F}, \widetilde{D}, \widetilde{G})$ introduced next.

As opposed to Theorem 5.3, which concerns the *existence* of a positive delay parameter, the results presented in this section provide sufficient conditions for pre-asymptotic stability *given* $T_{\min}$ and $T_{\max}$. As such, to simplify the notation, we do not explicitly parametrize the system $\widetilde{\mathcal{H}}$ under consideration. Letting $\widetilde{D} := D'$, where D' is given in (8), this leads to the jump map $\widetilde{G}$, where, for every $z \in \widetilde{D}$,

$$\widetilde{G}(z) := \begin{cases} \widehat{G}(x') \times [T_{\min}, T_{\max}] & \text{if } z \in D \times \{0_{n_2}\} \times \{-1\} \\ (x_1', \mu, 0_{n_2}, -1) & \text{if } z \in \mathbb{R}^n \times \mathbb{R}^{n_2} \times \{0\}. \end{cases}$$

Assuming the definition of F' in (7) to hold on $\mathbb{R}^n \times \mathbb{R}^{n_2} \times \mathbb{R}$, the flow map and flow set is defined as follows: $\widetilde{F} := F'$, and

$$\widetilde{C} := (C \times \{0_{n_2}\} \times \{-1\}) \cup (\widehat{C} \times [0, T_{\max}]),$$

where $\widehat{C}$ is a set satisfying

$$\widehat{C} \subset \{(x_1', x_2', \mu) : (x_1', x_2'), (x_1', \mu) \in C\}. \quad (14)$$

Unlike (6)-(7), the flow set and flow map are constructed using C and F themselves, respectively, and not their perturbations. The rationale behind this is that the generic structure of the perturbations in (5) could make stabilization a hopeless task.[7] As such, it is implicitly assumed that the structure of C and F gives rise to a meaningful $\widetilde{\mathcal{H}}$, in terms of realistically modeling delays. An additional change is with the flow set $\widetilde{C}$. It has a constraint in the form of $\widehat{C}$, which differs from (6). This is done so that a Lyapunov function for $\mathcal{H}$ can be used to quantify the evolution of (x_1', μ) during delays. Such a model is reasonable if $G(D) \subset \mathrm{cl}\, C$ and $C = C_1 \times C_2$ for some sets $C_1 \in \mathbb{R}^{n_1}$ and $C_2 \in \mathbb{R}^{n_2}$, in which case $\widehat{C}$ can be taken as $C_1 \times C_2 \times C_2$. In this case, selecting $\widehat{C}$ to be equal to the right-hand side of (14) captures all possible solutions in the presence of delays, while selecting it as a proper subset allows flexibility in the analysis. The latter is demonstrated in the proof of Theorem 7.2, which assesses asymptotic stability of sampled-data control with delays. As in Corollary 3.10, the following is also observed.

Lemma 6.1: The hybrid system $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions if and only if the set $\widehat{C}$ is closed and Assumption 2.2 holds.

B. Lyapunov-like Theorems for Asymptotic Stability in the Presence of Delays

In what follows, a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a *Lyapunov function candidate* for $\mathcal{H}$ if it is differentiable on an open set containing $\mathrm{cl}\, C$; cf. [1, Definition 3.16]. Given the closed set $\mathcal{A}$ of interest, we assume that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in C \cup D \cup G(D) \quad (15)$$

for some class-$\mathcal{K}_\infty$ functions α_1 and α_2. Note that differentiability of V implies that the bounds in (15) also hold on the boundary of C. We also make use of the following objects:

- The set $\mathcal{A}_1$, the projection of $\mathcal{A}$ onto $\mathbb{R}^{n_1}$.
- The projection Π from $\mathbb{R}^n \times \mathbb{R}^{n_2}$ onto $\mathbb{R}^n$. That is, for each $(x', \mu) \in \mathbb{R}^n \times \mathbb{R}^{n_2}$, $\Pi(x', \mu) = x'$.
- The set-valued mapping $F_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n_1}$, the projection of F onto $\mathbb{R}^{n_1}$. That is, for each $x \in \mathbb{R}^n$, $F_1(x)$ is the set of all y_1 such that $(y_1, y_2) \in F(x)$ for some $y_2 \in \mathbb{R}^{n_2}$.

During flows, the Lyapunov function candidate V is used to analyze the evolution of x' (when delays are inactive) and (x_1', μ) (when delays are active). It is also used to quantify jumps from x' to (x_1', μ) when delays are inactive. To analyze the evolution of x_2' during delays, an auxiliary Lyapunov-like function is used. The required properties for this function are given in Assumption 6.2. The conditions on the function are mild, and do not insist on descent properties, as the value of x_2' during a period of delay does not affect the value of x' after the delay expires. Importantly, although existence of this function is assumed, knowledge of its expression is not needed, as it does not play a role in the conditions of the upcoming results.

[7] For instance, perturbing F in (1) leads to disturbances on the differential equation $\dot{x}_p = f(x_p, u)$, which prohibits stabilization of the origin.

Assumption 6.2: There exists a function $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is differentiable on an open set containing $\mathrm{cl}(\Pi(\widehat{C}))$ such that

$$\tilde{\alpha}_1(|x_2|_{\mathcal{A}_2}) \leq \widetilde{V}(x) \leq \tilde{\alpha}_2(|x|_{\mathcal{A}_1 \times \mathcal{A}_2}) \quad \forall x = (x_1, x_2) \in \Pi(\mathrm{cl}(\widehat{C})) \tag{16}$$

for some class-$\mathcal{K}_\infty$ functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Furthermore, there exist $\tilde{\lambda} \in \mathbb{R}$ and a class-$\mathcal{K}_\infty$ function $\tilde{\alpha}$ such that

$$\langle \nabla \widetilde{V}(x), y \rangle \leq \tilde{\lambda} \widetilde{V}(x) \qquad \forall x \in \Pi(\widehat{C}), \forall y \in F(x), \tag{17}$$

$$\widetilde{V}(y) \leq \tilde{\alpha}(|x|_{\mathcal{A}}) \quad \forall x \in D, \forall y \in \Pi(\widehat{G}(x)) \cap \Pi(\mathrm{cl}(\widehat{C})). \tag{18}$$

Note that any given function V satisfying (15) also satisfies (16). In other words, (16) holds with $\widetilde{V} = V$, $\tilde{\alpha}_1 = \alpha_1$, and $\tilde{\alpha}_2 = \alpha_2$, since given $x = (x_1, x_2)$, $|x_2|_{\mathcal{A}_2} \leq |x|_{\mathcal{A}}$, and similarly, $|x|_{\mathcal{A}} \leq |x|_{\mathcal{A}_1 \times \mathcal{A}_2}$, as $\mathcal{A} \subset \mathcal{A}_1 \times \mathcal{A}_2$. Using the function $\widetilde{V}$, pre-asymptotic stability of $\widetilde{\mathcal{A}}$ for $\widetilde{\mathcal{H}}$ can be guaranteed, independent of $T_{\min}$ and $T_{\max}$, if a Lyapunov function candidate satisfies some descent conditions.

Theorem 6.3: Given a closed set $\mathcal{A}$, suppose that Assumption 6.2 holds. Let V be a Lyapunov function candidate for $\mathcal{H}$ satisfying (15) *for some class-$\mathcal{K}_\infty$ functions α_1 and α_2, and*

$$\langle \nabla V(x), y \rangle \leq -\, w(|x|_{\mathcal{A}}) \qquad \forall x \in C, \forall y \in F(x), \tag{19}$$

$$V(y) - V(x) \leq -\, w(|x|_{\mathcal{A}}) \qquad \forall x \in D, \forall y \in G(x), \tag{20}$$

for some continuous function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Moreover, suppose that

$$\langle \nabla_{x_1} V(x_1, \mu), y_1 \rangle \leq -w(|(x_1, \mu)|_{\mathcal{A}}) \quad \forall x_1, \mu, y_1 : \exists x_2, (x_1, x_2, \mu) \in \widehat{C}, y_1 \in F_1(x_1, x_2).$$

Then, the set $\widetilde{\mathcal{A}}$ in (13) *is stable for $\widetilde{\mathcal{H}}$. If, in addition, w is positive definite, then $\widetilde{\mathcal{A}}$ is globally pre-asymptotically stable for $\widetilde{\mathcal{H}}$.*

The proof of Theorem 6.3 is skipped, as it is similar to the proof of the next result, which is more involved. In fact, stability of $\widetilde{\mathcal{A}}$ follows in the exact same manner. Inspired by [1, Proposition 3.29], the second main result of the section, Theorem 6.4, relaxes some of the conditions in Theorem 6.3 (for example, (20)) by allowing V to increase during delays (respectively, jumps), as long as this increase is counteracted by a strong decrease during jumps (respectively, delays). This is shown in detail in the proof in Appendix A. Note that unlike Theorem 6.3, the conditions of Theorem 6.4 depend on $T_{\min}$ and $T_{\max}$.

Theorem 6.4: Given a closed set $\mathcal{A}$, suppose that Assumption 6.2 holds. Let V be a Lyapunov function candidate for $\mathcal{H}$ satisfying (15) *for some class-$\mathcal{K}_\infty$ functions α_1 and α_2, and* (19) *for some continuous function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Moreover, suppose that there exist $\lambda \in \mathbb{R}$ and a continuous positive definite function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\langle \nabla_{x_1} V(x_1, \mu), y_1 \rangle \leq \lambda V(x_1, \mu) \quad \forall x_1, \mu, y_1 : \exists x_2, (x_1, x_2, \mu) \in \widehat{C}, y_1 \in F_1(x_1, x_2), \tag{21}$$

$$V(y) \leq \varphi(V(x)) \quad \forall x \in D, \forall y \in G(x). \tag{22}$$

Consider the function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where

$$\gamma(r) := \max\{\exp(\lambda T_{\min}), \exp(\lambda T_{\max})\}\varphi(r) \quad \forall r \geq 0. \tag{23}$$

Then, the set $\widetilde{\mathcal{A}}$ in (13) *is stable for $\widetilde{\mathcal{H}}$ if $\gamma(r) \leq 1$ for all $r \geq 0$. If, in addition, either of the following conditions hold, then $\widetilde{\mathcal{A}}$ is globally pre-asymptotically stable for $\widetilde{\mathcal{H}}$:*

- *(D1) The function w is positive definite and $\gamma(r) < 1$ for all $r \geq 0$.*
- *(D2) There exists no eventually continuous and complete solution x of $\mathcal{H}$ originating from outside $\mathcal{A}$. Moreover, $\gamma(r) < 1$ for all $r \geq 0$.*
- *(D3) Every complete solution z of $\widetilde{\mathcal{H}}$ originating outside $\widetilde{\mathcal{A}}$ satisfies $\int_\Omega 1\, dt = \infty$, where*

$$\Omega := \{t : \exists (t, j) \in \mathrm{dom}\, z, \tau(t, j) = -1\}.$$

Moreover, w is positive definite.

Theorem 6.4 is illustrated in the next section. While similar results that allow V to grow during flows are possible, these are not included due to space limitations.

Remark 6.5: Theorems 6.3 and 6.4 can be applied to assess local pre-asymptotic stability by restricting the sets C and D to an ε-neighborhood of $\mathcal{A}$. In other words, C and D can be replaced with $C \cap (\mathcal{A} + \varepsilon\mathbb{B})$ and $D \cap (\mathcal{A} + \varepsilon\mathbb{B})$, respectively, for some $\varepsilon > 0$.

Remark 6.6: Under Assumption 2.2, when $\mathcal{A}$ is compact and $\widehat{C}$ is closed, as discussed in Remark 5.4, pre-asymptotic stability of $\widetilde{\mathcal{A}}$ is robust, due to Lemma 6.1. Moreover, it is uniform, in the sense that solutions of $\widetilde{\mathcal{H}}$ can be estimated by class-$\mathcal{KL}$ bounds, similar to (11).

VII. Examples and Applications

This section demonstrates some examples and applications.

A. Practical Stabilization of the Boost Converter with Switching Delays

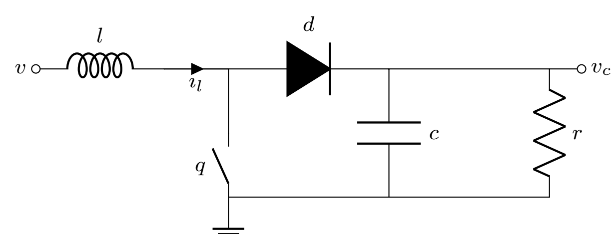

Fig. 1. Circuit diagram of the boost converter with capacitor c, diode d, voltage source v, inductor l, resistor r, and switch q.

We numerically evaluate the robustness properties of a boost converter under the hybrid control strategy in [28], with respect to switching delays. Figure 1 depicts a boost converter, which transfers energy from the supply to the load by increasing the magnitude of the input voltage. The voltage across the capacitor is denoted v_c, and the current through the inductor is denoted i_l. The converter draws power from the source and steps up the supply voltage by rapidly closing (to store energy in the inductor) and opening (to smoothly transfer energy to the capacitor supplying the load, modeled as a resistor) the switch. The diode allows the capacitor current to flow only towards the load and prevents discharging through the switch.

The overall dynamical model of the boost converter is a switched differential-algebraic equation with state $(v_c, \imath_l)$. Depending on the state of the switch (closed or open) and the diode (conducting or blocking), the converter can be in one of four different continuous modes of operation. Denoting by $q = 1$ (respectively, by $d = 1$) a closed switch (respectively, a conducting diode), and by $q = 0$ (respectively, by $d = 0$) an open switch (respectively, a blocking diode), the modes corresponding to $q = 1, d = 1$ and $q = 1, d = 0$ can be combined into a differential equation with continuous right-hand side. On the other hand, the combination of the remaining modes lead to a discontinuous differential equation. For robust global asymptotic stability, a Krasovskii regularization of the discontinuous vector field can be performed, leading to the switching differential-algebraic inclusion $(\dot{v}_c, \dot{\imath}_l) \in \widetilde{F}(v_c, \imath_l, q)$, where

$$\widetilde{F}(v_c, \imath_l, 0) := \begin{cases} \left\{\left(\frac{r\imath_l - v_c}{rc}, \frac{v - v_c}{l}\right)\right\} & \text{if } (v_c < v, \imath_l = 0) \\ & \text{or } \imath_l > 0 \\ \left\{-\frac{v_c}{rc}\right\} \times \left[\frac{v-v_c}{l}, 0\right] & \text{if } (v_c \geq v, \imath_l = 0) \end{cases} \tag{24}$$

and

$$\widetilde{F}(v_c, \imath_l, 1) := \left(-\frac{v_c}{rc}, \frac{v}{l}\right) \quad v_c \geq 0, \tag{25}$$

with v, r, l, and c positive. See [28] for the derivation.

When the discrete evolution of the switch q is governed by a feedback law, it is clear that these discrete transitions would be subject to delays in practice, due to the physical limitations of the electrical components realizing the switch. Hence, to study the robustness of the boost converter under the hybrid controller to be described, let $x_1 = (v_c, \imath_l)$ and $x_2 = q$. Let $p > 0$ be an arbitrary parameter. Given a desired setpoint $x_1^* := (v_c^*, (v_c^*)^2/(rv)) \in \mathbb{R}^2$ with $v_c^* > v$, let

$$\tilde{\gamma}_q(x_1) := \gamma_q(x_1) + K_q(v_c - v_c^*) \quad \forall x_1 \in \mathbb{R}^2, \forall q \in \{0,1\},$$

where $K_q \in (0, 2p/(rc))$, and $\gamma_q : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function[8] that depends on v_c^*, p, and the model parameters (r, l, c, and v), for each $q \in \{0,1\}$. The function γ_i has the property that if $x_1 \neq x_1^*$, $\tilde{\gamma}_q(x_1) \geq 0$ implies $\tilde{\gamma}_{1-q}(x_1) < 0$ and $\tilde{\gamma}_q(x_1) \leq 0$ implies $\tilde{\gamma}_{1-q}(x_1) > 0$, for each $q \in \{0,1\}$. An appropriate switching law is determined by the zero level sets of these functions, with the parameters K_0, K_1 tuning the shape of the level sets. Let

$$C_0 := \{(v_c, \imath_l, 0) : \imath_l \geq 0, \tilde{\gamma}_0(v_c, \imath_l) \leq 0\},$$
$$C_1 := \{(v_c, \imath_l, 1) : v_c \geq 0, \tilde{\gamma}_1(v_c, \imath_l) \leq 0\}.$$

Then, the closed-loop system is given by the data

$$\begin{aligned} C &= C_0 \cup C_1, \\ F(x) &= \widetilde{F}(x_1, q) \quad \forall x \in C, \\ D &= \cup_{q=0}^{1} \{(x_1, q) \in C_q : \tilde{\gamma}_q(x_1) = 0\}, \\ G(x) &= (x_1, 1-q) \quad \forall x \in D, \end{aligned} \tag{26}$$

[8]For the exact definitions of $\tilde{\gamma}_1, \tilde{\gamma}_2$, see [28, Eq. (8)] and [28, Eq. (9)], respectively, and replace p_{11} with p and p_{22} with $(pl)/c$.

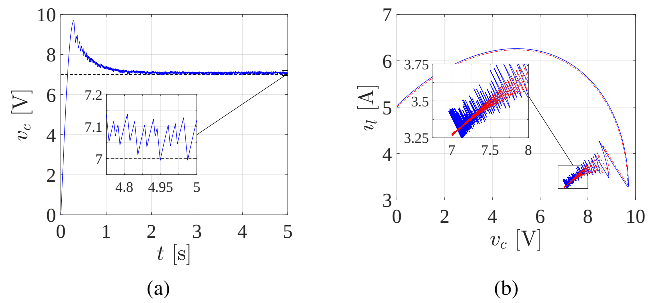

Fig. 2. Simulation results for the boost converter. (a) Evolution of v_c over ordinary time t with parameter $T = 0.005$. The black dashed line corresponds to $v_c^* = 7\,\text{V}$. (b) Range of two solutions (solid blue indicates $T = 0.005$, dotted red indicates $T = 0$) projected onto the v_c-$\imath_l$ plane.

where $\widetilde{F}$ is defined in (24)-(25).

An interesting feature of the hybrid system $\mathcal{H}$ with the data in (26) is that it has Zeno solutions, due to the fact that the switching functions satisfy $\tilde{\gamma}_0(x_1^*) = \tilde{\gamma}_1(x_1^*) = 0$. In particular, as the set $\mathcal{A} = \{x_1^*\} \times \{0,1\}$ is globally pre-asymptotically stable by [28, Th. IV.4], and because x_1^* is not an equilibrium point of (24) or (25), the unique maximal solutions from $(x_1^*, 0)$ and $(x_1^*, 1)$ are discrete (i.e., the domains are discrete sets). Since maximal solutions are complete [28, Proposition IV.1], these maximal solutions from $(x_1^*, 0)$ and $(x_1^*, 1)$ are Zeno.[9] Nevertheless, the hybrid controlled boost converter can tolerate switching delays in the sense of Theorem 5.3. For the hybrid system given in (26), Assumption 2.2 holds by [28, Lemma IV.3]. In addition, since $\mathcal{A}$ is globally pre-asymptotically stable, Assumption 4.2 is satisfied. Therefore, Theorem 5.3 applies.

We consider the simulation scenario in [28], where the boost converter steps up the supply voltage $v = 5\,\text{V}$ to the desired output voltage $v_c^* = 7\,\text{V}$. The physical parameters of the converter are given as $r = 3\,\Omega$, $l = 0.2\,\text{H}$ and $c = 0.1\,\text{F}$, leading to the desired setpoint $x_1^* = (7, 3.27)$. The controller parameters are selected as $p = c/2$, $K_0 = 0.05$, and $K_1 = 0.12$. The higher order model used to simulate the delayed boost converter is similar to the construction in Example 3.4, in that the perturbation function ρ is zero. The simulations[10] are performed using the Hybrid Equations Toolbox [29].

Figure 2a shows the evolution of the output voltage corresponding to the initial condition $(0, 5, 1, 1, -1)$, with switching delays up to 5 milliseconds, i.e. $T = 0.005$. The simulation is performed by choosing the length of jump delays randomly from the uniform distribution with support $[0, T]$. Despite the relatively large delay (the converter operates at the kHz range in the delay-free case, in the sense of the number of switches over a similar simulation horizon), the output voltage converges to the desired value in a practical sense, with a small error. The behavior of this solution can be compared to its delay-free counterpart in Figure 2b, where it can be observed that the solution remains in a neighborhood of the delay-free solution. Note that the distance between the two solutions is

[9]In fact, one can show that for every maximal solution, the time between consecutive jumps tends to zero.

[10]Code at https://github.com/HybridSystemsLab/DelayedBoostConverter

larger around the setpoint due to the increase in the switching frequency for the delay-free case.

TABLE I
PRACTICAL ERROR BOUND ε AND THE STEADY-STATE SWITCHING FREQUENCY CORRESPONDING TO THE LENGTH OF DELAYS.

Delay [ms]	Switching Frequency [Hz]	ε
0.625	1006.4	0.0286
1.25	568.5	0.0502
2.5	303.3	0.0933
5	160.0	0.1811
10	83.4	0.3518

To analyze the effect of the delay on the practical error bound ε and the average steady-state switching frequency, simulations are ran for various values of T, with the initial condition $(7, 3.27, 1, 0, -1)$. Since the x_1 component of this initial condition is precisely the desired setpoint, for each value of T, the error bound is estimated using this component. The length of jump delays are chosen randomly as before. It can be seen that the practical error scales linearly with the delay parameter T—it is estimated that the bound is roughly given by the formula[11] $35.6T$, thus validating the continuous dependence of the error ε on the delay parameter T. The importance of this result is underlined by the fact that practical stability is the best outcome in the presence of switching delays, as x_1^* is not an equilibrium point of (24) or (25).

B. Asymptotic Stabilization of Switched Systems with Switching Delays

Given $N, N_\circ \in \mathbb{N}$ and $\eta > 0$, consider the hybrid system $\mathcal{H}$ with state $x = (x_p, \tau_d, q) \in \mathbb{R}^{n_p} \times \mathbb{R} \times \mathbb{R}$ and the data

$$\begin{aligned} C &= \mathbb{R}^{n_p} \times [0, N_\circ] \times \{1, 2, \ldots, N\}, \\ F(x) &= \{A_q x_p\} \times [0, \eta] \times \{0\} \quad \forall x \in C, \\ D &= \mathbb{R}^{n_p} \times [1, N_\circ] \times \{1, 2, \ldots, N\}, \\ G(x) &= \{x_p\} \times \{\tau_d - 1\} \times \{1, 2, \ldots, N\} \quad \forall x \in D. \end{aligned} \tag{27}$$

The model here represents a switched system, where jumps correspond to mode switches, and at each mode q, the plant state x_p evolves according to the linear differential equation $\dot{x}_p = A_q x_p$, with A_q a real matrix of appropriate size. The timer state τ_d regulates switches such that for any solution x and any $(t, j), (s, i) \in \operatorname{dom} x$, $|j - i| \leq \eta |t - s| + N_\circ$. In the switched systems literature, this corresponds to what is said to be average dwell-time switching (after the first jump) with dwell time $1/\eta$ and offset $N_\circ$. The case $N_\circ = 1$ corresponds to dwell-time switching with dwell time $1/\eta$. At each jump time, the next mode is determined arbitrarily according to the difference inclusion $q^+ \in \{1, 2, \ldots, N\}$.

The next result provides conditions under which the set

$$\mathcal{A} = \{0_{n_p}\} \times [0, N_\circ] \times \{1, 2, \ldots, N\} \tag{28}$$

is globally asymptotically stable for $\mathcal{H}$ with the data in (27). Moreover, given the corresponding hybrid system $\widetilde{\mathcal{H}}$ in Section VI-A with $\widehat{C} = C \times \{1, 2, \ldots, N\}$, which represents switching delays on $\mathcal{H}$, using LMIs, we lay out constructive conditions for global asymptotic stability of the corresponding set $\widetilde{\mathcal{A}}$ in (13).

Theorem 7.1: Given the hybrid system $\mathcal{H} = (C, F, D, G)$ with the data in (27), for every $q \in \{1, 2, \ldots, N\}$, suppose that the matrix A_q is Hurwitz, and let P_q be a positive definite matrix such that $P_q A_q + A_q^\top P_q$ is negative definite. Consider the scalar η in (27). Then, the following statements hold:

1) *There exist $\tilde{\eta} > 0$, $\sigma > 0$, and $c \in [0, 1)$ such that*

$$\begin{aligned} P_q A_q + A_q^\top P_q < -\sigma \tilde{\eta} P_q \qquad & \forall q \in \{1, 2, \ldots, N\}, \\ P_{\tilde{q}} \leq c \exp(\sigma) P_q \quad & \forall q, \tilde{q} \in \{1, 2, \ldots, N\}. \end{aligned} \tag{29}$$

Moreover, the set $\mathcal{A}$ in (28) is globally asymptotically stable for $\mathcal{H}$ if there exist $\sigma > 0$ and $c \in [0, 1]$ such that (29) holds with $\tilde{\eta} = \eta$.

2) *For every $\tilde{\eta} > 0$ and every $\sigma > 0$, there exists $\lambda \in \mathbb{R}$ such that*

$$P_{\tilde{q}} A_q + A_q^\top P_{\tilde{q}} + \sigma \tilde{\eta} P_{\tilde{q}} \leq \lambda P_{\tilde{q}} \quad \forall q, \tilde{q} \in \{1, 2, \ldots, N\}. \tag{30}$$

Furthermore, given $\widehat{C} = C \times \{1, 2, \ldots, N\}$ and $T_{\min} = 0$, the set $\widetilde{\mathcal{A}}$ in (13) is globally asymptotically stable for $\widetilde{\mathcal{H}}$ if there exist $\sigma > 0$ and $c \in [0, 1)$ such that (29)-(30) hold with $\tilde{\eta} = \eta$ and $c \max\{1, \exp(\lambda T_{\max})\} < 1$ for some $\lambda \in \mathbb{R}$ satisfying (30).

Proof: Existence of $\tilde{\eta} > 0$, $\sigma > 0$, and $c \in [0, 1]$ satisfying (29) is obvious, due to the assumptions on A_q and P_q for each q. Now, suppose that there exists $\sigma > 0$ and $c \in [0, 1]$ such that (29) holds with $\tilde{\eta} = \eta$. Consider the Lyapunov function candidate $V(x) := \exp(\sigma \tau_d) x_p^\top P_q x_p$, which satisfies (15) for some class-$\mathcal{K}_\infty$ functions α_1 and α_2. Observe that the first LMI in (29) implies that V decreases during flows, i.e. (19) holds with positive definite w. Similarly, the second LMI in (29) implies that V is nonincreasing during jumps, i.e. (22) holds with $\varphi(r) = cr$ for all $r \geq 0$. Then, global asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$ follows by [1, Proposition 3.27]. This proves the first statement, and the second statement is obvious.

To prove asymptotic stability in the presence of delays, we use Theorem 6.4. The required conditions on V for Theorem 6.4 to hold have already been shown, and the inequality $c \max\{1, \exp(\lambda T_{\max})\} < 1$ corresponds to the function γ in (23) satisfying $\gamma(r) < 1$. Hence, Condition (D1) holds. It remains to show that there exists a function $\widetilde{V}$ satisfying Assumption 6.2. Indeed, Assumption 6.2 holds with $\widetilde{V} = V$. In particular, as discussed in Section VI, (16) holds with $\tilde{\alpha}_1 = \alpha_1$ and $\tilde{\alpha}_2 = \alpha_2$. In addition, (17)-(18) hold due to the descent properties of V. Thus, Theorem 6.4 applies. ■

The bounds on allowable delays stated in Theorem 7.1 can be conservative. Indeed, when $N_\circ = 1$, due to the nature of switched systems, global asymptotic stability with arbitrarily large delays can be observed by replacing η with $(1/\eta + T_{\min})^{-1}$. In fact, using such a strategy, it is straightforward to show that a large enough $T_{\min}$ stabilizes the set $\mathcal{A}$ in (28), even if it is not stable in the delay-free case. Asymptotic stability with delays is not obvious for controlled switched systems of the form $\dot{x}_p = (A_q + B_q K_{\tilde{q}}) x_p$, where

[11] T is in seconds; e.g. $T = 0.005$ implies delays up to 5 milliseconds.

there may be mismatches between the mode variables q and $\tilde{q}$ due to delays, but can also be addressed with Theorem 6.4.

C. Asymptotic Stabilization of Sampled-Data Control with Sample-and-Hold Delays

Consider the model of the sampled-data control system in (1)-(2) when f and κ are linear, namely, when

$$\begin{aligned} f(x_p, u) &= Ax_p + Bu \quad \forall x_p \in \mathbb{R}^{n_p}, \forall u \in \mathbb{R}^{n_c}, \\ \kappa(x_p) &= Kx_p \qquad\quad\;\; \forall x_p \in \mathbb{R}^{n_p}, \end{aligned} \tag{31}$$

for some A, B, and K. Let

$$A_f := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \text{ and } A_g := \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix} \tag{32}$$

be square matrices, where I and 0 denote the identity and zero matrices, respectively. Next, we show that if

$$\mathcal{A} = \{0_{n_p}\} \times [0, T_s] \times \{0_{n_c}\} \tag{33}$$

is globally asymptotically stable for the linear sampled-data control system, then there exists $T_{\max} > 0$ such that the set $\widetilde{\mathcal{A}}$ in (13) is globally asymptotically stable for the hybrid system $\widetilde{\mathcal{H}}$ (with $\widehat{C} = C \times \mathbb{R}^{n_c}$) representing delays on the input update event $u^+ = Kx_p$. For the definitions of $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{H}}$, see Section VI-A. For this task, we do not rely on Theorems 6.3 and 6.4. Instead, we consider the inequalities

$$\begin{aligned} Q^\top(s)\bar{P}(s)Q(s) - \bar{P}(T_s + s) < 0 \quad \forall s \in [0, \bar{s}], \\ R^\top(s)\bar{P}(s)R(s) - P < 0 \quad \forall s \in [0, \bar{s}], \end{aligned} \tag{34}$$

for a given symmetric P and $\bar{s} \geq 0$, where, for every $s \geq 0$,

$$\begin{aligned} \bar{P}(s) &:= \exp(A_f^\top(T_s - s))P\exp(A_f(T_s - s)), \\ Q(s) &:= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} \exp(-A_f s), \\ R(s) &:= \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \exp(A_f s). \end{aligned}$$

Theorem 7.2: Given the hybrid system $\mathcal{H} = (C, F, D, G)$ with the data in (1)-(2)*, suppose that the functions f and κ are given as in* (31) *for some A, B, and K. Suppose that the matrix $H := \exp(A_f T_s)A_g$ is Schur, where A_f and A_g are defined in* (32)*, and let P be a positive definite matrix such that $H^\top PH - P$ is negative definite. Then, the set $\mathcal{A}$ in* (33) *is globally asymptotically stable for $\mathcal{H}$, and there exists $\bar{s} > 0$ such that* (34) *holds. Furthermore, given $\widehat{C} = C \times \mathbb{R}^{n_c}$ and $T_{\min} = 0$, the set $\widetilde{\mathcal{A}}$ in* (13) *is globally asymptotically stable for $\widetilde{\mathcal{H}}$ if* (34) *holds with $\bar{s} = T_{\max}$.*

Proof: Asymptotic stability without delays is proved in [1, Example 3.26] by using the Lyapunov function

$$V(x) := \begin{bmatrix} x_p^\top & u^\top \end{bmatrix} \bar{P}(\tau_s) \begin{bmatrix} x_p \\ u \end{bmatrix} \quad \forall x = (x_p, \tau_s, u) \in C,$$

and existence of $\bar{s} > 0$ satisfying (34) follows from continuity and the fact that (34) simplifies to $H^\top PH - P < 0$ for $s = 0$. For asymptotic stability with delays, we exploit knowledge of V and come up with a Lyapunov function for $\widetilde{\mathcal{H}}$. Let

$$\widetilde{V}(z) := \begin{cases} V(x) & \text{if } \tau = -1 \\ \frac{V(\tilde{x}_p(x_p,\tau,u),\tau_s+\tau,\mu)+V(x_p,T_s+\tau_s,u)}{2} & \text{otherwise,} \end{cases}$$

where $\tilde{x}_p(x_p, \tau, u) := \begin{bmatrix} I & 0 \end{bmatrix} \exp(A_f\tau) \begin{bmatrix} x_p^\top & u^\top \end{bmatrix}^\top$. Note that $\widetilde{V}$ is smooth on a neighborhood of the closed set S, and positive definite with respect to the compact set $\widetilde{\mathcal{A}}$ on S, where $S := \widetilde{C} \cup \widetilde{D} \cup \widetilde{G}(\widetilde{D})$. In addition, it is constant during flows. This can be seen by noting that the term $\tilde{x}_p(x_p, u, \tau)$ corresponds to the plant state at the end of the delay period (due to the exponential) and making similar observations.

Now, we consider the case where $\widehat{C}$ is the set of all (x_1, x_2, μ) such that $(x_1, x_2) \in C$, $(x_1, \mu) \in C$, and given $x_1 = (x_p, \tau_s)$ and $x_2 = u$, $\tau_s \leq T_{\max}$ and

$$\mu = \begin{bmatrix} K & 0 \end{bmatrix} \exp(-A_f\tau_s) \begin{bmatrix} x_p \\ u \end{bmatrix}. \tag{35}$$

This is interpreted as the reachable set of

$$(\dot{x}_p, \dot{\tau}_s, \dot{u}, \dot{\mu}) = (Ax_p + Bu, 1, 0, 0) \quad \tau_s \in [0, T_{\max}]$$

from $\widehat{G}(D)$, in other words, the initial condition of the differential equation must satisfy $\mu(0) = Kx_p(0)$. Then, routine algebraic manipulations show that $V(y) - V(z)$ is negative for every $z \in (\widetilde{D} \cap \widetilde{C}) \backslash \widetilde{\mathcal{A}}$ and $y \in \widetilde{G}(z)$ if and only if (34) holds with $\bar{s} = T_{\max}$. By [1, Proposition 3.24], the set $\widetilde{\mathcal{A}}$ is globally asymptotically stable for the hybrid system with data $(\widetilde{C}, \widetilde{F}, \widetilde{D} \cap \widetilde{C}, \widetilde{G})$ if (34) holds.

For the general case of $\widehat{C} = C \times \mathbb{R}^{n_c}$, observe that there exists a finite $\eta \geq 0$ such that given any solution z of $\widetilde{\mathcal{H}}$, for every $(t, j) \in \operatorname{dom} z$ with $t + j \geq \eta$ and $\tau(t, j) \geq 0$,

$$\mu(t, j) = \begin{bmatrix} K & 0 \end{bmatrix} \exp(-A_f\tau_s(t, j)) \begin{bmatrix} x_p(t, j) \\ u(t, j) \end{bmatrix}$$

and $\tau_s(t, j) \leq T_{\max}$, which corresponds to the prior choice of $\widehat{C}$, defined via (35). Since η is finite, global asymptotic stability of $\widetilde{\mathcal{A}}$ for $\mathcal{H}$ for the case $\widehat{C} = C \times \mathbb{R}^{n_c}$ can be shown using 1) [1, Proposition 6.14] and global asymptotic stability of $\widetilde{\mathcal{A}}$ for the hybrid system with data $(\widetilde{C}, \widetilde{F}, \widetilde{D} \cap \widetilde{C}, \widetilde{G})$, with the prior choice of $\widehat{C}$. ■

Remark 7.3: The proof of Theorem 7.2 shows that even when Theorems 6.3-6.4 do not apply, knowledge about $\mathcal{H}$ or its stability certificate can be used to conclude asymptotic stability with delays: for the linear sampled-data control system, Theorems 6.3-6.4 apply only in trivial cases. Indeed, if (21) were true for a Lyapunov function V, along any solution with $\tau(0, 0) \geq 0$, $x_p(0, 0) = 0$ and $\mu(0, 0) = 0$, we would have $V(x_1(t, 0), \mu(t, 0)) \leq \exp(\lambda t)V(x_1(0, 0), \mu(0, 0)) = 0$ for all $(t, 0) \in \operatorname{dom} z$. Unless $T_{\max} = 0$, this implies that for an arbitrary constant input, the solution of $\dot{x}_p = Ax_p + Bu$ with zero initial condition is zero. Hence, B must be zero.

VIII. Conclusion

For hybrid systems $\mathcal{H}$ experiencing delays in their jumps, we constructed higher order models that depend on the length of delays and represent the effect of delayed jumps. Under mild conditions imposed on the delay-free system $\mathcal{H}$, it was shown that asymptotic stability of a given compact set is semiglobally practically robust with respect to the length of delays. Given minimum and maximum length of delays, sufficient conditions for asymptotic stability that allow a Lyapunov function for the delay-free system $\mathcal{H}$ to grow during delays were also presented. The obtained practical stability result was numerically validated on a hybrid controlled boost converter with switching delays. The higher order models and the sufficient Lyapunov conditions were used to derive constructive bounds on allowable delays for switched systems and sampled-data control.

The work here built the delay model on a couple of assumptions. It partitioned the state of $\mathcal{H}$ into a nondelayed and delayed component, and assumed that jumps due to the dynamics of $\mathcal{H}$ cannot occur during an active delay period. This selection is justified on the basis that it covers a wide variety of hybrid systems, including closed-loop systems with a single measurement/computational delay, while being simple enough to convey the main elements of the analysis in a clear fashion. It seems possible to extend this model where each of the n state components are subject to n distinct delays, and allow jumps due to the dynamics of $\mathcal{H}$ to occur during active delays, at the expense of added complexity. Such a model would allow the treatment of networked control systems with more than two agents, and provide formal robustness guarantees for multi-agent systems with protocols designed in the absence of delays.

Appendix A
Proofs

A. Proof of Proposition 3.7

For brevity, we only show that z satisfies the dynamics of $\mathcal{H}'$. If the set $I^j := \{t : (t,j) \in \operatorname{dom} z\}$ has a nonempty interior, by definition of $\operatorname{dom} z$, j must be even, so

$$z(t,j) = (x(t,j/2), 0_{n_2}, -1) \in C \times \{0_{n_2}\} \times \{-1\} \subset C'$$

for all $t \in \operatorname{int} I^j$, and

$$\begin{aligned}\dot z(t,j) = (\dot x(t,j/2), 0_{n_2}, 0) &\in F(x(t,j/2)) \times \{0_{n_2}\} \times \{0\} \\ &= F'(z(t,j)) \quad \text{for almost all } t \in \operatorname{int} I^j.\end{aligned}$$

Now assume $(t,j), (t,j+1) \in \operatorname{dom} z$. By definition of $\operatorname{dom} z$, if j is even, $(t,j/2), (t,j/2+1) \in \operatorname{dom} x$, so $x(t,j/2) \in D$. Consequently,

$$z(t,j) = (x(t,j/2), 0_{n_2}, -1) \in D \times \mathbb{R}^{n_2} \times \{-1\} \subset D',$$

$$\begin{aligned}z(t,j+1) = (x_1(t,j/2+1), x_2(t,j/2), x_2(t,j/2+1), 0) \\ = G'(z(t,j)).\end{aligned}$$

On the other hand, if j is odd, $(t,(j\pm1)/2) \in \operatorname{dom} z$, again by definition. Hence, $x(t,(j+1)/2) \in D$ and $t = t_{(j+1)/2}$, so

$$\begin{aligned}z(t,j) = ((&x_1(t,(j+1)/2), x_2(t,(j-1)/2)) \\ &, x_2(t,(j+1)/2), 0) \in \mathbb{R}^n \times \mathbb{R}^{n_2} \times \{0\} \subset D',\end{aligned}$$

$$z(t,j+1) = (x(t,(j+1)/2), 0, -1) = G'(z(t,j)).$$

Therefore, z is a solution to $\mathcal{H}$.

B. Proof of Proposition 4.3

Equivalence between pre-asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$ and $\mathcal{A}'$ for $\mathcal{H}$ is shown using Theorem C.1, by exploiting the relationship between the sets of solutions given in Lemma 3.8. An additional property, described in Lemma A.1, is also used.

Lemma A.1: The following statements are equivalent:

- *For all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$G((\mathcal{A} + \delta\mathbb{B}) \cap D) \subset \mathcal{A} + \varepsilon\mathbb{B}.$$

- *For all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$G'((\mathcal{A}' + \delta\mathbb{B}) \cap D') \subset \mathcal{A}' + \varepsilon\mathbb{B}. \tag{36}$$

Proof: Only sufficiency of the former statement for the latter is shown. Pick any $\varepsilon > 0$ and any $\delta \in (0, \min\{1, \varepsilon/2\})$ such that the inclusion $G((\mathcal{A}+\delta\mathbb{B}) \cap D) \subset \mathcal{A} + (\varepsilon/2)\mathbb{B}$ holds. Take any $z = (x', \mu, \tau) \in (\mathcal{A}' + \delta\mathbb{B}) \cap D'$ and $x' = (x_1', x_2')$. Since $\delta < 1$, if $\tau = -1$, then for all $(\tilde x_1', \tilde x_2', \tilde\mu) \in \widehat{G}(x')$,

$$(\tilde x_1', \tilde \mu) \in G(x') \subset \mathcal{A} + (\varepsilon/2)\mathbb{B} \text{ and } \tilde x_2' = x_2' \in \mathcal{A}_2 + \delta\mathbb{B},$$

which, since $\delta < \varepsilon/2$, implies that $G(z) \subset \mathcal{A}' + \varepsilon\mathbb{B}$, by definition of P. Otherwise, if $\tau = 0$, then $(x_1', \mu) \in \mathcal{A} + \delta\mathbb{B}$, which implies that $G'(z) \subset \mathcal{A}' + (\varepsilon/2)\mathbb{B}$, hence (36) holds. ■

Let $\widetilde D := D \times \{0_{n_2}\} \times \{-1\} \subset D'$, and given $z \in \mathcal{S}_{\mathcal{H}'}$, let

$$\begin{aligned}E(z) &:= \operatorname{dom} z \\ &\quad \setminus \{(t,j) \in \operatorname{dom} z : (t,j-1) \in \operatorname{dom} z, z(t,j-1) \in \widetilde D\}.\end{aligned}$$

Suppose Assumption 4.2 holds. Then, for all $\varepsilon > 0$, there exists $\eta \in (0, \min\{\varepsilon, 1\})$ such that every $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{A} + \eta\mathbb{B})$ satisfies $|x(t,j))|_{\mathcal{A}} \le \varepsilon$ for all $(t,j) \in \operatorname{dom} x$. In turn, there exists $\delta \in (0,\eta)$ such that $G'((\mathcal{A}' + \delta\mathbb{B}) \cap D') \subset \mathcal{A}' + \eta\mathbb{B}$ by Lemma A.1. Take any $z = (x', \mu, \tau) \in \mathcal{S}_{\mathcal{H}'}(\mathcal{A} + \delta\mathbb{B})$. Then, either $\tau(0,0) = -1$, or $(0,1) \in \operatorname{dom} z$ and $\tau(0,1) = -1$. Hence, by Lemma 3.8,

$$z \in \mathcal{S}_{\mathcal{H}'}(\mathcal{A}' + \delta\mathbb{B}) \implies |z(t,j)|_{\mathcal{A}'} \le \varepsilon \quad \forall (t,j) \in E(z). \tag{37}$$

Similarly, using Lemma A.1, Lemma 3.8, and Assumption 4.2, there exists $\sigma > 0$ such that every $z \in \mathcal{S}_{\mathcal{H}'}(\mathcal{A} + \sigma\mathbb{B})$ with unbounded $E(z)$ satisfies

$$\lim_{\substack{t+j\to\infty \\ (t,j)\in E(z)}} |z(t,j)|_{\mathcal{A}'} = 0. \tag{38}$$

Now, observe that the sets $G'(\widetilde D)$ and $\widetilde D$ are disjoint. Moreover, under Assumption 4.2, by Lemma A.1, for all $\varepsilon > 0$, there exists $\delta > 0$ such that (36) holds, which implies that $\mathcal{A}'$ is pre-asymptotically stable for the hybrid system with data $(\varnothing, F', \widetilde D, G')$. Therefore, by Theorem C.1, it follows that $\mathcal{A}'$ is pre-asymptotically stable for $\mathcal{H}'$.

In the other direction, if $\mathcal{A}'$ is pre-asymptotically stable for $\mathcal{H}'$, again by Theorem C.1, 1) for all $\varepsilon > 0$, there exists $\delta > 0$ such that (37) holds, and 2) there exists $\sigma > 0$ such that (38) holds. Using Lemma 3.8, it can be shown that these properties imply Assumption 4.2, completing the proof.

C. Proof of Theorem 6.4

Let α_1 and α_2 be class-$\mathcal{K}_\infty$ functions such that (15) holds. Pick a class-$\mathcal{K}_\infty$ function α_3 that upper bounds the continuous positive definite function φ. For every $r \geq 0$, let

$$\begin{aligned} \alpha_4(r) &:= \max\{r, c\alpha_3(r)\}, \\ \alpha_5(r) &:= \max\{r, \alpha_1^{-1}(c\alpha_4(\alpha_2(r)))\}. \end{aligned} \tag{39}$$

Take any function $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is differentiable on an open set containing $\mathrm{cl}(\Pi(\widehat{C}))$ such that (16)-(18) hold for some $\widetilde{\lambda} \in \mathbb{R}$ and class-$\mathcal{K}_\infty$ functions $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\tilde{\alpha}$. For every $r \geq 0$, let

$$\begin{aligned} \tilde{\alpha}_3(r) &:= \max\{\alpha_5(r), \tilde{\alpha}_1^{-1}(\tilde{c}\tilde{\alpha}(\alpha_5(r)))\}, \\ \tilde{\alpha}_4(r) &:= \max\{r, \tilde{\alpha}_1^{-1}(c\tilde{\alpha}_2(r))\}. \end{aligned} \tag{40}$$

1) Stability: Take any $z = (x', \mu, \tau) \in \mathcal{S}_{\widetilde{\mathcal{H}}}$ and let E be the set of all $(t,j) \in \mathrm{dom}\, z$ such that $\tau(t,j) \geq 0$, denoting hybrid times with active delays. For any $(t,j) \in \mathrm{dom}\, z$, let

$$x(t,j) := \begin{cases} (x_1'(t,j), \mu(t,j)) & \text{if } (t,j) \in E \\ x'(t,j) & \text{if } (t,j) \notin E. \end{cases} \tag{41}$$

Note that by (21), the function V can grow along x during delays (i.e., on E) by a factor of $c := \max\{1, \exp(\lambda T_{\max})\}$, in a worst-case sense. Also, observe that

$$x(0,0) \notin \mathrm{cl}(C) \cup D \implies |x(0,1)|_{\mathcal{A}} = |x(0,0)|_{\mathcal{A}}, \tag{42}$$

since, by definition of $\widetilde{C}$ and $\widetilde{D}$, if $x(0,0) \notin \mathrm{cl}(C) \cup D$, necessarily, $\tau(0,0) = 0$ and $\mathrm{dom}\, z = \{(0,0), (0,1)\}$.

Suppose that $x(0,0) \in \mathrm{cl}(C) \cup D$. Consider the case where $\tau(0,0) = -1$. Then, $(t,j) \in E$ if and only if j is odd. Pick any $(t,j) \in \mathrm{dom}\, z \backslash E$ and let $\{t_i\}_{i=0}^{j+1}$ be the sequence satisfying

$$\mathrm{dom}\, z \cap ([0,t] \times \{0,1,\ldots,j\}) = \cup_{i=0}^{j} [t_i, t_{i+1}] \times \{i\}.$$

Since j is even, by (19) and the fact that $\gamma(r) \leq 1$ for all $r \geq 0$,

$$V(x(t,j)) \leq V(x(0,0)) - \sum_{i=0}^{j/2} \int_{t_{2i}}^{t_{2i+1}} w(|x(s,2i)|_{\mathcal{A}})\, ds. \tag{43}$$

Therefore, $V(x(t,j)) \leq V(x(0,0))$ for all $(t,j) \in \mathrm{dom}\, z \backslash E$. Recalling the class-$\mathcal{K}_\infty$ function α_3 upper bounding φ, it follows by (22) and (21) that $V(x(t,j)) \leq c\alpha_3(V(x(0,0)))$ for all $(t,j) \in \mathrm{dom}\, E$. Hence, by definition of α_4 in (39),

$$V(x(t,j)) \leq \alpha_4(V(x(0,0))) \quad \forall (t,j) \in \mathrm{dom}\, z. \tag{44}$$

Otherwise, if $\tau(0,0) \geq 0$, then $V(x(t,0)) \leq cV(x(0,0)$ for all $(t,0) \in \mathrm{dom}\, z$. Combining this with (44), which holds when $\tau(0,0) = -1$, it follows by (15) that

$$\begin{aligned} \alpha_1(|x(t,j)|_{\mathcal{A}}) &\leq V(x(t,j)) \leq c\alpha_4(V(x(0,0))) \\ &\leq c\alpha_4(\alpha_2(|x(0,0)|_{\mathcal{A}})) \quad \forall (t,j) \in \mathrm{dom}\, z, \end{aligned} \tag{45}$$

provided $x(0,0) \in \mathrm{cl}(C) \cup D$. Hence, by (42), (45), and (39),

$$|x(t,j)|_{\mathcal{A}} \leq \alpha_5(|x(0,0)|_{\mathcal{A}}) \quad \forall (t,j) \in \mathrm{dom}\, z. \tag{46}$$

Now, consider the evolution of $\widetilde{V}$ along x' during delays, and let $\tilde{c} := \max\{1, \exp(\widetilde{\lambda} T_{\max})\}$. Take any $(s,j) \in E$ such that $(s, j-1) \in \mathrm{dom}\, z \backslash E$, which implies $j \geq 1$. If the inclusion $(x'(s,j), \mu(s,j)) \in \mathrm{cl}\, \widehat{C}$ holds, then by (18) and (17), $\widetilde{V}(x'(t,j)) \leq \tilde{c}\tilde{\alpha}(|x'(s,j-1)|_{\mathcal{A}})$ for every t such that $(t,j) \in E$. By (46) and the lower bound in (16), this implies that for every $(t,j) \in E$,

$$\tilde{\alpha}_1(|x_2'(t,j)|_{\mathcal{A}_2}) \leq \widetilde{V}(x'(t,j)) \leq \tilde{c}\tilde{\alpha}(\alpha_5(|x(0,0)|_{\mathcal{A}})), \tag{47}$$

since $x'(s,j-1) = x(s,j-1)$ by (41). Else, the inequality

$$\begin{aligned} |x_2'(s,j)|_{\mathcal{A}_2} = |x_2'(s,j-1)|_{\mathcal{A}_2} &\leq |x'(s,j-1)|_{\mathcal{A}} \\ &= |x(s,j-1)|_{\mathcal{A}} \leq \alpha_5(|x(0,0)|_{\mathcal{A}}) \end{aligned}$$

holds by (46), as $x_2'(s,j) = x_2'(s,j-1)$ by definition of $\widehat{G}$ and $x'(s,j-1) = x(s,j-1)$ due to (41). Combining this bound with the one in (47), it follows that

$$\begin{aligned} |x_2'(t,j)|_{\mathrm{cl}\,\mathcal{A}_2} = |x_2'(t,j)|_{\mathcal{A}_2} &\leq \tilde{\alpha}_3(|x(0,0)|_{\mathcal{A}}) \\ &\forall (t,j) \in E \cap (\mathbb{R}_{\geq 0} \times \{1,2,\ldots\}), \end{aligned} \tag{48}$$

where $\tilde{\alpha}_3$ is given in (40). When $(0,0) \in E$, two cases are of interest. If $(x'(0,0), \mu(0,0)) \in \mathrm{cl}\, \widehat{C}$, then by (17), for every $(t,0) \in E$, $\widetilde{V}(x'(t,0)) \leq \tilde{c}\widetilde{V}(x'(0,0))$. Else, there exists no $(t,0) \in \mathrm{dom}\, E$ with $t > 0$. This, coupled with (16) and the definition of $\tilde{\alpha}_4$ in (40), implies that

$$\begin{aligned} |x_2'(t,0)|_{\mathrm{cl}\,\mathcal{A}_2} = |x_2'(t,0)|_{\mathcal{A}_2} &\leq \tilde{\alpha}_4\left(|x'(0,0)|_{\mathcal{A}_1 \times \mathcal{A}_2}\right) \\ &\forall (t,0) \in E. \end{aligned} \tag{49}$$

Pick any $\varepsilon > 0$ and take $\eta \in (0,1)$ satisfying

$$\max\{\alpha_5(\eta), \tilde{\alpha}_3(\eta), \tilde{\alpha}_4(\eta)\} \leq \varepsilon/2. \tag{50}$$

Take $\delta \in (0, \eta/2)$. Given $z = (x', \mu, \tau) \in \mathcal{S}_{\widetilde{\mathcal{H}}}(\widetilde{\mathcal{A}} + \delta\mathbb{B})$, define E and x as before. Observe that since $\delta < 1$, the inequalities $|x(0,0)|_{\mathcal{A}} \leq \delta$ and $|x_2'(0,0)|_{\mathcal{A}_2} \leq \delta$ hold. By definition of x in (41), the former also implies $|x_1'(0,0)|_{\mathcal{A}_1} \leq \delta$ and therefore $|x'(0,0)|_{\mathcal{A}_1 \times \mathcal{A}_2} \leq \eta$. Also, by (46) and (50),

$$|x(t,j)|_{\mathcal{A}} \leq \varepsilon/2 \quad \forall (t,j) \in \mathrm{dom}\, z. \tag{51}$$

Hence, $|z(t,j)|_{\widetilde{\mathcal{A}}} \leq \varepsilon/2$ for all $(t,j) \in \mathrm{dom}\, z \backslash E$. Similarly, by (48)-(49) and (50), $|x_2'(t,j)|_{\mathrm{cl}\,\mathcal{A}_2} \leq \varepsilon/2$ for all $(t,j) \in E$. By (51), it follows that $|z(t,j)|_{\widetilde{\mathcal{A}}} \leq \varepsilon$ for all $(t,j) \in E$. Consequently, $\widetilde{\mathcal{A}}$ is stable.

2) Attractivity: Pick a complete $z = (x', \mu, \tau) \in \mathcal{S}_{\widetilde{\mathcal{H}}}$ and define E and x as before. Since $\widetilde{\mathcal{A}}$ is stable, it suffices to only consider the case when $z(0,0) \notin \widetilde{\mathcal{A}}$. Also note that since z is complete, $x(0,0) \in \mathrm{cl}(C) \cup D$, and the set $\mathrm{dom}\, z \backslash E$ is unbounded as the length of delays are finite. Hence, without loss of generality, assume $\tau(0,0) = -1$. Suppose Condition (D3) holds. Then, the Lebesgue measure of the set

$$\{t : \exists (t,j) \in \mathrm{dom}\, z \backslash E\} \tag{52}$$

is infinite. Thus, using (43), which holds when $\tau(0,0) = -1$ and $(t,j) \in \mathrm{dom}\, z \backslash E$, standard contradiction arguments in Lyapunov analysis can be employed to show that

$$\lim_{\substack{t+j \to \infty \\ (t,j) \in \mathrm{dom}\, z \backslash E}} V(x(t,j)) = \lim_{\substack{t+j \to \infty \\ (t,j) \in \mathrm{dom}\, z \backslash E}} |x(t,j)|_{\mathcal{A}} = 0. \tag{53}$$

On the other hand, if (D2) holds, then E is unbounded. Thus, letting $\{t_j\}_{j=1}^{\infty}$ be the jump times of z,

$$V(x(t_{j+2}, j+2)) \leq \gamma(V(x(t_j, j))) \quad \forall j \in \{0, 2, \dots\}, \tag{54}$$

where $t_0 = 0$. This can again be used to deduce (53) by noting that $(t_j, j) \in \operatorname{dom} z/E$ for every $j \in \{0, 2, \dots\}$, as V is nonincreasing along x on $\operatorname{dom} z/E$. Now, suppose Condition (D1) holds. If z is not eventually continuous, E must be unbounded, so (54) and therefore (53) follows. If z is eventually continuous, then (53) follows directly, as the Lebesgue measure of the set in (52) must be infinite. In other words, as long as at least one of (D1)-(D3) hold, (53) holds.

Now, for any sequence $\{(t_i, j_i)\}_{i=0}^{\infty} \in \operatorname{dom} z \backslash E$ such that $\lim_{i\to\infty} t_i + j_i = \infty$, by (46) and (53)

$$\limsup_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = \lim_{i\to\infty} \sup_{t+j\geq t_i+j_i} \alpha_5(|x(t_i, j_i)|_{\mathcal{A}}) = 0,$$

and similarly, by (48) and (53)

$$\limsup_{\substack{t+j\to\infty \\ (t,j)\in E}} |x_2'(t,j)|_{\operatorname{cl}\mathcal{A}_2} = \lim_{i\to\infty} \sup_{t+j\geq t_i+j_i} \tilde{\alpha}_3(|x(t_i, j_i)|_{\mathcal{A}}) = 0.$$

Hence, $\lim_{t+j\to\infty} |z(t,j)|_{\widetilde{\mathcal{A}}} = 0$. Finally, note that for any solution $z = (x', \mu, \tau)$ of $\widetilde{\mathcal{H}}$, the function $(t,j) \mapsto |z(t,j)|_{\widetilde{\mathcal{A}}}$ is bounded on $\operatorname{dom} z$ by (46), (48), and (49). Therefore, $\widetilde{\mathcal{A}}$ is globally pre-asymptotically stable.

Appendix B
On Maximum Allowable Delays

In [30, Lemma 2.7], it is shown that when $\mathcal{H}$ satisfies Assumption 2.2 and consecutive jumps are not possible, given a complete and bounded solution, the elapsed time between jumps is uniformly bounded away from zero. Below, we generalize this result uniformly over the set solutions in a semiglobal sense, with the additional stability assumption. This justifies the structure of the jump map G_T' in (9) when consecutive jumps are not allowed, as discussed in Section III.

Proposition B.1: Suppose that Assumptions 2.2 and 4.2 hold, the set $\mathcal{A}$ is compact, and $G(D) \cap D$ is empty. Then, the basin of pre-attraction $\mathcal{B}_{\mathcal{A}}^p$ of $\mathcal{A}$ is open, and for every compact set $K \subset \mathcal{B}_{\mathcal{A}}^p$, there exists $\eta > 0$ with the following property: for every solution x of $\mathcal{H}$ originating from K, the ordinary time elapsed between jumps is lower bounded by η. That is, for every solution x of $\mathcal{H}$ originating from K and every $j \in \{1, 2, \dots\}$ such that $(t,j), (t, j+1) \in \operatorname{dom} x$ for some t, $\sup\{s - t : \exists (t,j), (s,j) \in \operatorname{dom} x\} \geq \eta$.

Proof: That $\mathcal{B}_{\mathcal{A}}^p$ is open follows from [1, Proposition 7.4]. Take any compact set $K \subset \mathcal{B}_{\mathcal{A}}^p$. By [1, Th. 7.12], the set

$$\mathcal{R}(K) := \{x(t,j) : x \in \mathcal{S}_{\mathcal{H}}(K) \text{ and } (t,j) \in \operatorname{dom} x\}$$

is bounded, so $F(\operatorname{cl}(\mathcal{R}(K)) \cap C) \subset \varepsilon\mathbb{B}$ for some $\varepsilon > 0$, as F is locally bounded and outer semicontinuous relative to the closed set C. Hence, for every $x \in \mathcal{S}_{\mathcal{H}}(K)$, $|\dot{x}(t,j)| \leq \varepsilon$ for almost all $(t,j) \in \operatorname{dom} x$. Let $\Delta := \operatorname{cl}(\mathcal{R}(K)) \cap D$. Since Δ is compact, Condition (A3) implies $G(\Delta)$ is compact. Morever, since $G(\Delta)$ and Δ are disjoint, the Hausdorff distance between them must be positive. The remainder of the proof is similar to that of [30, Lemma 2.7]. ■

Appendix C
Stability Analysis Through Event Exclusions

The following theorem is used to extend the stability properties of $\mathcal{H}$ to $\mathcal{H}'$. The main idea is to show that if solutions of the hybrid system $\mathcal{H}'$ are "stable" when those points corresponding to events (a particular type of jump) are excluded and the discrete-time model representing the events is stable, then $\mathcal{H}'$ should be stable. This result is closely related to the stability analysis of hybrid systems through limited events [31], but does not require stabily properties to be global or the number of event occurrences to be finite. In the statement of Theorem C.1, the events are defined as jumps from the set $\widetilde{D} \subset D$, and the set $E(x) \subset \operatorname{dom} x$ excludes all (t,j) corresponding to these events. If the set $\mathcal{A}$ is compact, Conditions (S2) and (S4) hold when G is locally bounded and outer semicontinuous relative to $\widetilde{D}$, $G(\mathcal{A} \cap \widetilde{D}) \subset \mathcal{A}$, and there exists $J \in \mathbb{N}$ such that the solutions of the hybrid system with data $(\varnothing, F, \widetilde{D}, G)$ jump at most J times.

Theorem C.1: Given the hybrid system $\mathcal{H} = (C, F, D, G)$ and a set $\widetilde{D} \subset D$, for any solution x of $\mathcal{H}$, let

$$E(x) := \operatorname{dom} x \\ \backslash\{(t,j) \in \operatorname{dom} x : (t, j-1) \in \operatorname{dom} x, x(t, j-1) \in \widetilde{D}\}.$$

Then, a set $\mathcal{A}$ is stable for $\mathcal{H}$ if and only if the following hold:

(S1) For all $\varepsilon > 0$, there exists $\delta > 0$ such that every solution x of $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in E(x)$.

(S2) The set $\mathcal{A}$ is stable for the hybrid system with data $(\varnothing, F, \widetilde{D}, G)$.

Moreover, $\mathcal{A}$ is pre-asymptotically stable for $\mathcal{H}$ if and only if Conditions (S1)-(S2) and the following hold:

(S3) There exists $\sigma > 0$ such that for every solution x of $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq \sigma$, if $E(x)$ is unbounded, then

$$\lim_{\substack{t+j\to\infty \\ (t,j)\in E(x)}} |x(t,j)|_{\mathcal{A}} = 0. \tag{55}$$

(S4) The set $\mathcal{A}$ is pre-asymptotically stable for the hybrid system with data $(\varnothing, F, \widetilde{D}, G)$.

Proof: Necessity is obvious for both conclusions. For sufficiency, let $\mathcal{D} := (\varnothing, F, \widetilde{D}, G)$. Pick any $\eta > 0$. By (S2), there exists $\varepsilon \in (0, \eta)$ such that $y \in \mathcal{S}_{\mathcal{D}}(\mathcal{A} + \varepsilon\mathbb{B})$ implies $|y(0,j)|_{\mathcal{A}} \leq \eta$ for all $(0,j) \in \operatorname{dom} y$. In turn, by (S1), there exists $\delta > 0$ such that $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{A} + \delta\mathbb{B})$ implies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in E(x)$. Now, observe that given any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{A} + \delta\mathbb{B})$ and any $(t,j) \in \operatorname{dom} x \backslash E(x)$, there exists $i < j$ such that $(t,i) \in E(x)$, and $x(t,k) \in \widetilde{D}$ for all $k \in \{i, i+1, \dots, j-1\}$. Consequently, $|x(t,k)|_{\mathcal{A}} \leq \eta$ for all $k \in \{i, i+1, \dots, j-1\}$. As such, $\mathcal{A}$ is stable for $\mathcal{H}$.

Next, take $\varepsilon > 0$ such that every complete $y \in \mathcal{S}_{\mathcal{D}}(\mathcal{A} + \varepsilon\mathbb{B})$ satisfies $\lim_{j\to\infty} |y(0,j)|_{\mathcal{A}} = 0$. By (S1) and (S3), there exists $\varsigma > 0$ such that $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{A} + \varsigma\mathbb{B})$ implies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in E(x)$, and if $E(x)$ is unbounded, then (55) holds. Take any complete $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{A} + \varsigma\mathbb{B})$. If $E(x)$ or $\operatorname{dom} x \backslash E(x)$ is bounded, it is trivial to show $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$. Suppose the opposite and pick any sequence $\{(t_i, j_i)\}_{i=0}^{\infty} \in \operatorname{dom} x$ satisfying $\lim_{i\to\infty} t_i + j_i = \infty$. If this sequence is in $E(x)$, the statement $\lim_{i\to\infty} |x(t_i, j_i)|_{\mathcal{A}} = 0$ follows directly. Else, if the sequence is in $\operatorname{dom} x\backslash E(x)$, by the same argument used in proving stability, there exists a sequence $\{(s_i, k_i)\}_{i=0}^{\infty} \in E(x)$ satisfying $\lim_{i\to\infty} s_i + k_i = \infty$ with the following property: for each $i \in \mathbb{N}$, there exists $y_i \in \mathcal{S}_{\mathcal{D}}(x(s_i, k_i))$ such that $x(t_i, j_i) \in \operatorname{rge} y_i$, where rge denotes the range of a function. Noting that $\lim_{i\to\infty} |x(s_i, k_i)|_{\mathcal{A}} = 0$, one can invoke the stability property in (S2) to conclude $\lim_{i\to\infty} |x(t_i, j_i)|_{\mathcal{A}} = 0$. On the other hand, if the sequence has elements in both $E(x)$ and $\operatorname{dom} x\backslash E(x)$, by passing to appropriate subsequences, the same conclusion can be reached. Therefore, $\mathcal{A}$ is pre-asymptotically stable for $\mathcal{H}$. ∎

Corollary C.2: The set $\mathcal{A}$ is globally pre-asymptotically stable for $\mathcal{H}$ if and only if the following hold:

- *Condition (S1) holds. Moreover, for every solution x of $\mathcal{H}$, the function $(t, j) \mapsto |x(t, j)|_{\mathcal{A}}$ is bounded on $E(x)$, and if $E(x)$ is unbounded, then (55) holds.*
- *The set $\mathcal{A}$ is globally pre-asymptotically stable for the hybrid system with data $(\varnothing, F, D, G)$.*

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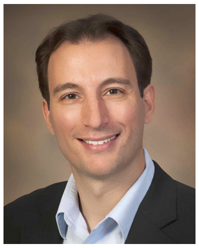
Berk Altın (SM'13, M'17) received the B.S. degree in mechatronics from Sabancı University, Istanbul, Turkey, in 2011, and the M.S. and Ph.D. degrees in electrical engineering: systems and the M.S. degree in mathematics from the University of Michigan, Ann Arbor, MI, USA, in 2013, 2016, and 2016, respectively.

From 2011 to 2016, he was a Fulbright Fellow with the University of Michigan. He is currently a Post-Doctoral Researcher with the Hybrid Systems Laboratory, University of California, Santa Cruz, CA, USA. His research interests include hybrid systems, model predictive control, iterative learning control, repetitive processes, and multidimensional systems, with applications in cyber-physical systems, power systems, robotics, and additive manufacturing.



Ricardo G. Sanfelice (SM'01, M'08, S'13) received the B.S. degree in electronics engineering from the Universidad Nacional de Mar del Plata, Buenos Aires, Argentina, in 2001, and the M.S. and Ph.D. degrees in electrical and computer engineering from the University of California, Santa Barbara, CA, USA, in 2004 and 2007, respectively.

In 2002, he joined the Center for Control, Dynamical Systems, and Computation, University of California, Santa Barbara, CA, USA. In 2007 and 2008, he was a Postdoctoral Associate in the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, USA. He visited the Centre Automatique et Systemes, Ecole de Mines de Paris, Paris, France, for four months. He is currently a Professor of electrical and computer engineering at the University of California, Santa Cruz, CA, USA. His research interests include modeling, stability, robust control, observer design, and simulation of nonlinear and hybrid systems with applications to power systems, aerospace, and biology.

Prof. Sanfelice received the 2013 SIAM Control and Systems Theory Prize, the National Science Foundation CAREER Award, the Air Force Young Investigator Research Award, and the 2010 IEEE Control Systems Magazine Outstanding Paper Award.