Relaxing Fundamental Assumptions in Iterative Learning Control

by

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To my parents Mehmet and Ufuk, my brother İlke, my sister (in law) Ayşim, my niece İlkim, and last but not least, my love Jenny

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Let us meet again... In a hundred years.

Berk Altın
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LIST OF ABBREVIATIONS

ILC Iterative learning control

SISO single-input single-output

LTI linear time invariant

2D two dimensional

AM additive manufacturing

LMD laser metal deposition

1D one dimensional

MIMO multiple-input multiple-output

BIBO bounded-input bounded-output

AILC adaptive ILC

HOIM higher order internal model

$\mathcal{L}_1$ **AC** $\mathcal{L}_1$ adaptive control

MRAC model reference adaptive control

$\mathcal{L}_1$**-ILC** $\mathcal{L}_1$ AC-ILC

BIBS bounded-input bounded-state

LTI-ILC LTI output feedback based ILC

PID proportional-integral-derivative

LQR linear quadratic regulation

DRP differential repetitive process

LTV linear time varying

0-i.s. zero initial state

ABSTRACT

Relaxing Fundamental Assumptions in Iterative Learning Control

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Co-Chairs: Kira L. Barton and Jessy W. Grizzle

Iterative learning control (ILC) is perhaps best decribed as an open loop feedforward control technique where the feedforward signal is learned through repetition of a single task. As the name suggests, given a dynamic system operating on a finite time horizon with the same desired trajectory, ILC aims to iteratively construct the inverse image (or its approximation) of the desired trajectory to improve transient tracking. In the literature, ILC is often interpreted as feedback control in the iteration domain due to the fact that learning controllers use information from past trials to drive the tracking error towards zero. However, despite the significant body of literature and powerful features, ILC is yet to reach widespread adoption by the control community, due to several assumptions that restrict its generality when compared to feedback control. In this dissertation, we relax some of these assumptions, mainly the fundamental invariance assumption, and move from the idea of learning through repetition to two dimensional systems, specifically repetitive processes, that appear in the modeling of engineering applications such as additive manufacturing, and sketch out future research directions for increased practicality: We develop an $\mathcal{L}_1$ adaptive feedback control based ILC architecture for increased robustness, fast convergence, and high performance under time varying uncertainties and disturbances. Simulation

studies of the behavior of this combined $\mathcal{L}_1$-ILC scheme under iteration varying uncertainties lead us to the robust stability analysis of iteration varying systems, where we show that these systems are guaranteed to be stable when the ILC update laws are designed to be robust, which can be done using existing methods from the literature. As a next step to the signal space approach adopted in the analysis of iteration varying systems, we shift the focus of our work to repetitive processes, and show that the exponential stability of a nonlinear repetitive system is equivalent to that of its linearization, and consequently uniform stability of the corresponding state space matrix.

CHAPTER 1

Introduction

ILC is best decribed as an open loop feedforward control technique where the feedforward signal is “learned” through repetition of a single task. As the name suggests, given a dynamic system operating on a finite time horizon with the same desired trajectory, ILC aims to iteratively construct the inverse image (or its approximation) of the desired trajectory to improve transient tracking. In the literature, ILC is often interpreted as feedback control in the iteration domain due to the fact that learning controllers use information from past trials to drive the tracking error towards zero. In an abstract manner, let $P : U \to Y$, where U is the space of admissible inputs and Y is the space of outputs. Assuming that P is known and there are no exogenous inputs affecting the output, the classical ILC problem can be stated as that of finding a controller C that maps the input history $u_0, u_1, \dots, u_{k-1} \in U$ to the current input u_k such that the output $y_k = Pu_k$ converges to a desired reference r in the image of P, or a small neighborhood of it, as the iteration number $k \to \infty$. In most cases, C is designed to consider the information from only the previous iteration, thus giving rise to the name *first order ILC*. More generally, ILC can be viewed as a special class of *repetitive processes* (also known as multipass processes earlier in the literature) [2]; that is systems where the dynamics at trial k is a function of the output history $y_0, y_1, \dots, y_{k-1}$. In ILC, the trial domain dynamics are induced on the input u_k through the design of an update law as the process is inherently a static or memoryless repetitive process.

Relatively speaking, ILC is a young but well established area of research. The roots of ILC can be traced back to the works of Uchiyama [3], published in Japanese, with Arimoto's 1984 paper [4] widely accepted as among the first[1] formal works on ILC, although some earlier ideas that align with the ILC paradigm have appeared in the 1970s [7]. Despite the significant body of literature, ILC is yet to reach widespread adoption by the control community: Wherein the search terms "robust control" and "'adaptive control" generate over 10,000 and 11,000 papers, respectively, "iterative learning control" generates merely 465 papers[2] in `ieeeXplore.ieee.org` [8]. Apart from the fact that ILC is much younger than conventional control disciplines, one reason for this disparity is that ILC is subject to several assumptions that restrict its generality when compared to feedback control. Yet, ILC is a very powerful technique that has the potential to equip modern systems with enhanced capabilities: It is hypothesized in [9] that ILC is loosely based on human learning. This hypothesis is supported by the findings of Zhang et al. [10], and Zhou et al. [11]. This potential is further underlined by the fact that as opposed to some other intelligent[3] control techniques, ILC is simple, easy to implement, and more importantly has proven stability and convergence conditions guaranteeing perfect tracking.

[1]Craig [5], and Casalino and Bartolini [6] have published two other similar papers in the same year independently of Arimoto, although these two papers have not attracted the same level of attention.

[2]As of the end of 2005. As of April 5, 2016, the search terms "robust control" and "'adaptive control" generate over 58,319 and 71,632 papers, respectively, "iterative learning control" generates 2,714 papers.

[3]In [12], the authors argue that ILC is an intelligent control technique since it "uses conventional control methods to solve lower level control problems", and "attempts to build upon and enhance the conventional control methodologies to solve new challenging control problems", based on a report by Panos Antsaklis [13].

1.1 The Invariance Assumption in ILC

ILC offers several advantages over feedback control such as improved transient response, potential for "noncausal"[4] operation, and the ability to compensate for repetitive effects, without resorting to high gain feedback. The standard assumption in classical ILC is that of iteration invariance, of

1. The time interval $[0, T]$ in which the system operates,
2. The plant P,
3. The desired reference r,
4. The exogenous disturbance d,
5. The initial condition $x(0)$.

Here P may be thought of as an open loop stable plant, or the input-output relationship of a closed loop stabilized plant. Although unrealistic, the above assumptions lead to simple yet powerful results. For instance, consider the following single-input single-output (SISO) discrete linear time invariant (LTI) system

$$\begin{aligned} x_k(t+1) &= Ax_k(t) + Bu_k(t), \quad x_k(0) = x_0, \\ y_k(t) &= Cx_k(t), \end{aligned}$$

for all $t \in \{0, 1, \dots, T\}$ and $k \in \mathbb{N}$, where $x_k(t) \in \mathbb{R}^n$ is the state vector, $u_k(t) \in \mathbb{R}$ is the input, $y_k(t) \in \mathbb{R}$ is the output, and A, B, C are appropriately sized real matrices. Assume that the system has relative degree 1. Take the *update law*

$$u_{k+1}(t) = u_k(t) + l(r(t+1) - y(t+1)),$$

[4] Of course, ILC is subject to causality in the strict sense as we can only process information from past trials. However, the operator that we use to process this data can be noncausal in the sense that the input $u_{k+1}(t)$ can depend on $u_{k-l}(t+\tau)$ for some $t, \tau > 0$ and $l \in \{0, 1, \dots, k\}$.

where $r(t) \in \mathbb{R}$ is the reference signal. Then $y_k(t) \to r(t)$ as $k \to \infty$ if and only if $|1 - lCB| < 1$, or equivalently if $lCB \in (0, 2)$. Hence, the sign of the first nonzero Markov parameter is all that is necessary to construct the feedforward inverse that achieves perfect tracking, since the the inequality can be satisfied by decreasing $|l|$ provided we choose $\text{sgn}(l) = \text{sgn}(CB)$.

The idea of learning an input signal u_∞ that would achieve perfect tracking is a very attractive feature of classical ILC. However, in practice, perfect tracking could be an infeasible, inachievable, or undesired objective. For instance, in the presence of measurement noise, a more reasonable strategy would be to design controllers that converge to a neighborhood of the origin. If P is subject to some uncertainty, the perfect tracking objective can result in update laws that violate certain robustness criteria and result in unstable algorithms. Alternatively, in some contexts (e.g. pick and place robotic applications), a subset of $[0, T]$ could be of interest rather than the whole interval [14–17].

1.2 The Feedback-Learning Analogy

As stated before, the paradigm of ILC can be readily connected to feedback control by selecting the iteration as the dependent variable as opposed to time. A more direct treatment of this issue is discussed in several papers, where the converged ILC system is found to be equivalent to a feedback controller for causal algorithms [18–22]. To further underline the similarities between feedback and learning, let us have a closer look at the definition of feedback control. Broadly speaking, the objective in feedback control, or control theory in general, is to manipulate the input of a system in a way so that the output behaves as desired. In today's automated world, control is vital for the proper operation of many devices and offers the development of new technologies which would have otherwise been impossible. Control actively shapes society by enabling modern machinery to be fast, efficient, consistent, and reliable. A nice way of interpreting control theory is that control engineers seek to find procedures that would solve given classes of problem objectives dynamically as opposed to finding solutions themselves: In the classical tracking problem for a plant P, it may indeed be possible to compute explicitly an input u that solves the problem. However, this fundamentally relies on the unrealistic assumptions that:

1. P is known perfectly and is invariant in time.

2. P is not subject to exogenous disturbances.

For instance, if in addition to the above assumptions, we assume P to be invertible, we may uniquely select $u = P^{-1}r$ for a given reference r. However, by synthesizing a feedback relationship, we can compensate for exogenous disturbances and variations in P over time. Thus, roughly speaking, we can claim that control engineers design controllers that "learn" the desired task asymptotically *in time*. It is the job of the engineer to find controllers that achieve the best performance in terms of trade-offs imposed by closing the loop, that are sufficiently general, flexible, robust, and easily implementable.

In terms of the terminology used in describing feedback control, ILC "learns" the desired task asymptotically in the iteration domain. As such, it is the job of the engineer to find controllers that achieve the best performance in terms of trade-offs imposed by closing *the iteration loop*, that are sufficiently general, flexible, robust, and easily implementable. In practice, much as in conventional feedback control, by synthesizing a recurrence relationship, we can compensate for violations of certain assumptions listed above. For instance, iterative learning of an optimal feedforward action as opposed to analytical computation can compensate for changes in P over time (iteration), with the converged error $e_\infty = 0$ given that P varies slowly. This idea can be interestingly linked to more general methods such as the proof of the

Picard-Lindelöf theorem, where the solution of the ordinary differential equation is constructed through an iterated sequence, or more strongly to inversion techniques that rely on Picard-like iterations [23]. Similar ideas are also used in system identification [24–27]; for example in [28] power iteration like methods are used to estimate the $\mathcal{H}_\infty$ norm of a system.

Regardless, in an increasingly automated and smart world, it is desirable that the assumptions are relaxed *in theory* in order for ILC to find use in a broader application space and be more widely adopted. Especially when a perceived advantage of ILC over other intelligent control methods is simplicity [9], it is necessary that the focus of ILC is shifted from "control" to "learning".

1.3 About Repetitive Processes

The feedback in the iteration domain interpretation of ILC is a powerful analogue that paves the way into repetitive processes, which are two dimensional (2D) dynamic systems that are characterized by sequences of finite passes who contribute to the evolution of the future passes. These systems appear in applications such as additive manufacturing (AM), wherein products are built via layer by layer deposition; a specific example being laser metal deposition (LMD) [29, 30]. An LTI repetitive process can be described as follows:

$$\dot{x}_{k+1}(t) = Ax_{k+1}(t) + By_k(t) + B_u u_{k+1}(t),$$
$$y_{k+1}(t) = Cx_{k+1}(t) + Dy_k(t) + D_u u_{k+1}(t),$$

for all $t \in [0, T]$ and $k \in \mathbb{N}$, where A, B, C, D, B_u, D_u are continuous real matrices of appropriate size. Here, the output at layer k acts as a forcing function on the dynamics of layer $k + 1$. In the simplest case of a perfect AM process, the layer to layer dynamics would be a perfect integrator, i.e. $D = I$ and $B = 0$. Ignoring initial conditions and fixing $u_k = u$, one way to interpret this process is that the filter defined by (A, B, C, D) is applied recursively to find y satisfying

$$y(s) = (C(sI - A)^{-1}B + D)y(s) + (C(sI - A)^{-1}B_u + D_u)u(s).$$

The interpretation discussed here shows that recursive algorithms for one dimensional (1D) dynamic systems fall within the field of repetitive processes. Of course, one problem that arises here is whether the process converges in a stable fashion to the equilibrium signal y.

1.4 Problem Statement

Moore, Chen, Ahn, and Xu [31–33] have identified possible directions for future ILC research as listed below:

- Nonlinear ILC: Nonlinear update laws have not been extensively researched in ILC, save for adaptive learning laws for locally Lipschitz plants.
- Spatiotemporal dynamical systems: ILC theory for partial differential equations is not well understood. The practical infeasibility of having continuous measurements point out to different directions for research.
- Performance analysis: Linear ILC is relatively mature and hence performance oriented methodologies, design limitations, guidelines are increasingly important.
- Fractional order dynamics: Fractional systems are an interesting new area of research, examples of such systems can be found with polymers, piezo materials, silicon gel etc.
- Network controlled systems and cooperative ILC: Consensus building, control

under uncertain communication topologies, intermittent sensing and actuation are some problems associated with these areas.

In addition to the above, based on our previous discussion we pose the following questions:

- Can the iteration invariance assumptions on $P, d, x(0)$ be relaxed, while still maintaining the powerful features of classical ILC?
- Is it possible for a system to "learn" when r is iteration varying, under certain conditions?
- Is it possible to shift the dependence from time to another variable? Can iteration varying time intervals be considered?
- What are the limits of achievable performance and robustness bounds for iteration invariant or varying systems?

The inspiration for the first question is drawn mainly from additive manufacturing. While every repetitive system would be subject to variance in P, d and $x(0)$, additive manufacturing is an application area in which the change from iteration to iteration can be quite high and uncertain. Repetitive process theory [2] provides another good motivation for this area in terms of the layer by layer material deposition procedure, as discussed before. The second question targets applications that do not involve repetitive operation in the classical sense; a potential application for this scenario is flight control, where gain scheduling and adaptive feedback control is common. The third question aims to generalize the fundamental objective from the typical tracking problem. For instance, for a robot that involves repetitive motion we may wish to minimize the time elapsed to complete each action, or some other performance measure. On a higher level, we may wish that the robot "learns" a different action from previous actions; that is we expect that the robot extrapolates a new task based on prior learned tasks in its memory[5]. Finally, we wish to maximize performance and robustness bounds for iteration varying and invariant systems to provide a *systematic and practical* framework, in order to encourage the widespread adoption of ILC for different practical applications.

1.5 Contributions and Organization of the Dissertation

This dissertation addresses some of the problems raised in Section 1.4 by moving from a robust ILC framework towards stability analysis of nonlinear repetitive processes. The specific problems we focus on are listed as follows:

1. Can the iteration invariance assumptions on $P, d, x(0)$ be relaxed, while still maintaining the powerful features of classical ILC?

2. Is it possible for a system to "learn" when r is iteration varying, under certain conditions?

3. What are the limits of achievable performance and robustness bounds for iteration invariant or varying systems?

4. Nonlinear update laws have not been extensively researched in ILC, save for adaptive learning laws for locally Lipschitz plants. What are necesssary and sufficient conditions for stability?

After providing a brief technical overview of ILC in Chapter 2, and presenting prior literature as it relates to our problem statement in Chapter 3, along with a categorical review of general ILC literature, the original work that led to this dissertation is presented in Chapters 4 to 6.

[5] This has been explored previously in [34, 35] for the output tracking problem. Extending this approach to higher level learning remains an open question.

Chapter 4 presents an original robust ILC framework for precision motion control applications, motivated by the plant invariance problem. This framework uses $\mathcal{L}_1$ adaptive feedback control to decrease parameter uncertainty and thereby reduce conservativeness in the learning algorithms to obtain better performance. The integration of the feedback control strategy into a learning framework raises questions of stability and design trade-offs which are addressed throughout the chapter. Through simulations on the model of a flexure bearing based nanopositioner, it is shown that $\mathcal{L}_1$ adaptive control provides up to an order of magnitude improvement in transient tracking, in addition to significantly increasing predictability of the system under sudden parameter changes from iteration to iteration. This Chapter is partially based on [36–38].

The simulation scenario of abrupt parameter changes from iteration to iteration naturally leads to the analysis of iteration varying systems, which is discussed in Chapter 5. The specific problem tackled in the chapter is the robust stability and performance of ILC systems violating the restrictive invariance assumption in an abstract vector space setting. It is shown through basic mathematical analysis that robust monotonic update laws lead to stable systems when the iteration varying plant uncertainties are within the uncertainty set, and that the performance of the invariant certain system can be recovered if the uncertainties are convergent along the iteration axis. Some comments are made on the design trade-offs between predictability and nominal performance, and an optimization approach is suggested for the update law design for iteration varying uncertain systems. The findings of the chapter are verified via simulations and experiments on a linear motion control stage. This Chapter is partially based on [39, 40].

As a natural next step to the holistic signal space approach adopted in the analysis of iteration varying systems, the focus of our work shifts to repetitive processes in Chapter 6. As the existing literature on repetitive processes is predominantly on

LTI systems, and repetitive processes in AM applications such as laser metal deposition [29, 30] appear as nonlinear models, the chapter analyzes exponential stability properties of nonlinear time varying repetitive processes from a local perspective. New definitions of stability that depend on initial state sequences as well as the initial output are developed. The exponential stability criterion of LTI systems is extended to the time varying case. This spectral radius criterion is connected to nonlinear systems through local stability analysis, which is conducted partially by using abstract Lyapunov functionals. Our main result shows that exponential stability of a nonlinear system and its linearization is equivalent, which can be guaranteed by making sure that the relevant state space matrix is uniformly Schur over all time. We use this result to analyze local stability of Picard iterations with nonconstant initial states, as well as nonlinear ILC algorithms. Simulation studies are conducted on the model of an actuated Van der Pol oscillator with time varying damping; it is shown that an ILC algorithm using the second derivative of the error can solve the problem of uniformly tracking a sinusoidal reference, without any stabilizing feedback. This Chapter is partially based on [41, 42].

Concluding remarks, along with plans and suggestions for future research directions are given in Chapter 7. For a more compact presentation, additional technical material for Chapters 4 and 6 are given in Appendices A and B, respectively.

CHAPTER 2

Technical Overview of ILC

In this chapter, we will present a brief overview of ILC. We start with the classical ILC problem, which will be first formulated in an abstract setting to keep some generality as we saw in Chapter 1: Let $P : U \to Y$ be a mapping where U is the space of admissible inputs and Y is the space of outputs. When P is known and there are no exogenous inputs affecting the output, the problem can be stated as that of finding a controller C that maps the input history $u_0, u_1, \ldots, u_{k-1} \in U$ to the current input $u_k \in U$ such that the output $y_k = Pu_k$ converges to a desired reference r in the image of P, as the iteration number $k \to \infty$. In most cases, C is designed to consider the information from only the previous iteration, thus giving rise to the name *first order ILC*. Algorithms that consider multiple iterations on the other hand, are called *higher order ILC*.

Now let us consider the case where U and Y are Banach spaces equipped with suitable norms, consistent with the approach in [7] and [2]. We base this assumption on the fact that Banach spaces are the natural settings of contraction mapping based ILC problems, which rely on the celebrated fixed point theorem. This is hardly a restriction as we can assume most spaces that we work on in practice to be complete[6]. For instance, $\mathcal{L}_p$ and l_p spaces, which provide a general framework in time driven

[6]Completeness is not even a vital property and is just needed to ensure that a fixed point exists. The contraction condition is sufficient to guarantee that we converge to a limit point in the completion of the space, as we will discuss in the following pages.

dynamic systems, are well known to be complete normed spaces. Our motivation in considering the problem in a Banach space setting is twofold: First, we would like to keep the analysis simple, general, and intuitive. Second, we wish not to limit the discussion to dynamic systems in the classical sense; that is, systems defined by ordinary differential equations. Indeed, as mentioned in Chapter 1, there are other areas of research in control and related fields that bear significant resemblance to the problem of iteratively constructing an input to track a desired reference.

To develop the notions of stability, convergence, and boundedness for ILC problems, let us give some basic definitions. Of course, in ILC, such concepts should all be defined over the iteration domain. Hence we assume that the plant P is well posed in the sense of basic input-output stability; that P is either a bounded operator, or in the case that P represents an unstable dynamic system on a time interval $[0, T]$, the escape time is larger than T[7]. For a rigorous study of these issues, we define the spaces $U^\omega \triangleq \prod_{k\in\mathbb{N}} U$ and $Y^\omega \triangleq \prod_{k\in\mathbb{N}} Y$. An element x in these spaces will be defined so that x_k denotes the kth coordinate. Alternatively, we define $x \triangleq (x_0, x_1, \dots)$ to be a mapping from $\mathbb{N}$, the set of nonnegative integers, to U or Y, where each x_k can be an element of U, Y. In addition, we introduce the following definitions where the spaces X and V are in $\{U, Y\}$. We will use $\|.\|$ to denote vector and induced operator norms in the relevant spaces.

Definition 2.1. Let x be an element of X^ω. We say x is bounded if $\|x\| < \infty$ and unbounded otherwise, where $\|x\| \triangleq \sup_{k\in\mathbb{N}} \|x_k\|$.

The definition of boundedness is in essence the familiar notion of uniform boundedness, renamed to reflect the repetitive nature of the ILC problem. Readers would also note that U^ω and Y^ω are not normed spaces since our definition of the norm entails the possibility of unbounded elements. However, this is merely a formality and

[7] This is more of a theoretical assumption. In practice, we would most likely be working on a stable or stabilized system.

will not affect our analysis as any truncated vector in these spaces has a finite norm. This is akin to the definition of input-output stability via the extended space $\mathcal{L}_{pe}$. Boundedness will not be studied in detail in this chapter since due to the discrete nature of the ILC problem convergence implies boundedness; we will say that $x \in X^\omega$ converges to an element $\bar{x} \in X$ if $x_k \to \bar{x}$.

Definition 2.2 (Asymptotic Stability)**.** Let $H : X \to X$. An iterative system defined by the equality $x_{k+1} = Hx_k$ for all $k \in \mathbb{N}$ is *asymptotically stable* if for all $\epsilon > 0$ there exists $\delta > 0$, and a neighborhood $\bar{X}$ of the unique fixed point $\bar{x}$ of H such that

$$\|x_0\| < \delta \implies \|x_k\| < \epsilon, \quad \forall k \in \mathbb{N},$$

and x converges to $\bar{x}$ for all $x_0 \in \bar{X}$.

Asymptotic stability is usually an insufficient condition in practice since asymptotically stable systems may exhibit large transient growth before beginning convergent behavior [9, 43]. A stronger notion is *monotonic convergence*, which is ubiquitous in contraction mapping based ILC. In fact, this is one of the strongest stability notions that we have for the problem since convergence of any kind implies boundedness by virtue of the discrete nature of the problem.

Definition 2.3 (Monotonic Convergence)**.** An asymptotically stable system is called monotonically convergent if there exists a neighborhood $\bar{X}_\mathrm{m}$ of the fixed point such that for all $x_0 \in \bar{X} \cap \bar{X}_\mathrm{m}$,

$$\|\bar{x} - x_{k+1}\| \leq \gamma \|\bar{x} - x_k\|, \quad \forall k \in \mathbb{N}.$$

Before proceeding with the analysis, we will recall a fundamental result from metric spaces; Banach's celebrated fixed point (or contraction mapping) theorem. The theorem plays a very important role in classical ILC as most of the fundamental convergence results can be proven with little to no effort through the formulation of a contraction. By treating the theorem seperately, we will see that the design of stable, monotonically convergent iterative learning controllers becomes almost a trivial matter even in an abstract setting.

Theorem 2.1 (Banach Fixed Point)**.** *Let $H : X \to X$ be a contraction mapping on a complete normed space X; i.e. there exists $\gamma < 1$ such that*

$$\|Hx - Hy\| \leq \gamma \|x - y\|, \quad \forall x, y \in X.$$

Then, H has a unique fixed point $\bar{x} = H\bar{x}$. Moreover, for any $x_0 \in X$, the sequence generated by $x_{k+1} = Hx_k$ for all $k \in \mathbb{N}$ converges to $\bar{x}$ with $\|\bar{x} - x_{k+1}\| \leq \gamma \|\bar{x} - x_k\|$.

The contraction mapping theorem can be likened to the small gain theorem [44,45], where the constant γ can be thought of as the gain factor. The proof of the theorem shows that the contraction condition $\gamma < 1$ ensures that the sequence $\{x_k\}_{k=0}^{\infty}$ is Cauchy, and therefore convergent.

2.1 Contraction Mapping Based ILC

With the developments given before, consider

$$y_k = Pu_k + d, \quad \forall k \in \mathbb{N}, \tag{2.1}$$

where $y_k \in Y$ is the output, $u_k \in U$ is the input, $d \in Y$ is the exogenous signal that includes the feedback control response, disturbance, noise, and the effect of initial conditions, and P is the bounded linear input-output operator. The objective is to find an ILC update law such that the error e defined by $e_k \triangleq r - y_k$ for all $k \in \mathbb{N}$, where the reference r is in the image of P, converges to a small neighborhood of 0.

Hence, consider the linear first order update law

$$u_{k+1} = Qu_k + Le_k, \quad \forall k \in \mathbb{N}, \tag{2.2}$$

where Q and L are linear operators, and u_0 is arbitrary. Before stating the convergence of the algorithm, we will need the following result from functional analysis.

Lemma 2.1. *Let $H : X \to X$ be a bounded linear operator, where X is complete and $\|H\| < 1$. Then $(I - H)^{-1} = \sum_{i=0}^{\infty} H^k$, where I is the identity operator on X.*

Proof. Let $x \in X$ be arbitrary and take $m, n \in \mathbb{N}$. Without loss of generality, assume $m > n$. Then $\sum_{i=0}^{m} H^k x - \sum_{i=0}^{n} H^k x = H^{n+1} \sum_{i=0}^{m-n-1} H^k x$, so

$$\left\| \sum_{i=0}^{m} H^k x - \sum_{i=0}^{n} H^k x \right\| \leq \|H\|^n \frac{1 - \|H\|^{m-n}}{1 - \|H\|} \|x\| \leq \|H\|^n \frac{1}{1 - \|H\|} \|x\|.$$

Therefore, the partial sums $(\sum_{i=0}^{n} H^k x)_{n \in \mathbb{N}}$ form a Cauchy and hence convergent sequence since $\|H\| < 1$ and X is complete. In other words, $\sum_{i=0}^{\infty} H^k$ is an operator on X. Moreover, direct computation shows that

$$(I - H) \left(\sum_{i=0}^{\infty} H^k \right) x = \left(\sum_{i=0}^{\infty} H^k \right) (I - H) x = x.$$

Thus, it follows that $(I - H)^{-1} = \sum_{i=0}^{\infty} H^k$. ■

The above lemma guarantees the existence of the inverse operator, which will be used to find the fixed point of the recurrence relation. We are now ready state the monotonic convergence theorem for linear systems.

Theorem 2.2 (Monotonic Convergence)**.** *The ILC system described by* (2.1) *with the update law* (2.2) *is monotonically convergent if $\|Q - LP\| \leq \gamma < 1$. The fixed point of the system is $\bar{u} = (I - Q + LP)^{-1} L(r - d)$, where $\|.\|$ is the induced norm on U.*

Proof. Substituting (2.1) into (2.2) yields $u_{k+1} = (Q - LP)u_k + L(r - d)$. Define the mapping $H : U \to U$ such that $Hu = (Q - LP)u + L(r - d)$ for all $u \in U$. Then, given arbitrary $u, v \in U$, $Hu - Hv = (Q - LP)(u - v)$ from linearity so H is a contraction if and only if $\|Q - LP\| < 1$. It follows from the fixed point theorem that the system is monotonically convergent if $\|Q - LP\| \leq \gamma < 1$. In other words, we have $\|\bar{u} - u_{k+1}\| \leq \gamma \|\bar{u} - u_k\|$ for some $\bar{u}$. To find the fixed point, consider the equality $\bar{u} = (Q - LP)\bar{u} + L(r - d)$ and rearrange terms to solve for $\bar{u}$. ■

Corollary 2.1 (Converged Error)**.** *If the ILC system described by (2.1) with the update law (2.2) satisfies the monotonic convergence condition of Theorem 2.2, the converged error of the system is given by*

$$\bar{e} = r - (P\bar{u} + d) = (I - P(I - Q + LP)^{-1}L)(r - d).$$

Proof. Follows from the fact that the error map from the input $u_k \mapsto r - (Pu_k + d)$ is continuous since P is bounded. ■

Theorem 2.2 gives a sufficient condition for monotonic convergence[8] of linear ILC systems. There are a number of converses [46, 47] to the contraction mapping theorem, so it can be regarded as a necessary condition in a certain sense. One such converse [48] states that in a T_1 space[9] we can find a metric so that the recurrence relation is a contraction. This implies that asymptotically stable systems are also monotonically convergent in a certain metric. However, this metric may not capture the desired properties from an engineering standpoint, so Theorem 2.2 is still a very significant result: In general, we would most likely want to see monotonic convergence in the 2 norm or the sup norm, or their analogues. More importantly, the necessary and sufficient condition for checking stability requires the computation of the spectral

[8]Note that monotonic convergence is considered for the input here. This condition can be modified to ensure monotonic convergence in the error.

[9]In point set topology, a T_1 space is a space in which given any two distinct elements x and y, we can find a neighborhood of x that does not contain y, and vice versa.

radius of the linear operator $Q - LP$, which is a nontrivial task when the underlying input space is infinite dimensional [2, page 44]. On the other hand, generalizing the monotonic convergence theorem to even nonlinear P is a straightforward matter when we trade the condition $\|Q - LP\| < 1$ with $\|(Q - LP)u - (Q - LP)v\| < \gamma\|u - v\|$ for arbitrary $u, v \in U$, where $\gamma < 1$. Finally, the condition *can always be satisfied* in any normed space as shown in the trivial result below:

Proposition 2.1 (Stabilizability)**.** *Given $\gamma \in [0, 1)$ and linear $P : U \to Y$, there exists a pair (Q, L) such that $\|Q - LP\| = \gamma < 1$.*

Proof. Let $Q, \bar{L}$ be arbitrary bounded operators. Select $L = Q\bar{L}$. Then,

$$Q - LP = Q(I - \bar{L}P),$$

so $\|Q - LP\| \leq \|\bar{Q}\|\|(I - \bar{L}P)\|$ and one can satisfy the norm condition by decreasing the gain of Q. ■

The specific case where $L = Q\bar{L}$ is called Q *filtering* in the literature; which reformulates the update law as $u_{k+1} = Q(u_k + Le_k)$. The Q filter is often used as a robustifying measure at the expense of *perfect tracking* (zero asymptotic errors), which can most intuitively be seen in an $\mathcal{H}_2$ setting: Most physical plants have near $\pm 180^\circ$ phase uncertainty in the high frequency region, which can be gain stabilized by selecting Q as a low pass filter with bandwidth similar to the passband of P [49]. Often $\bar{L}$ is selected as the approximate inverse of P to ensure low converged errors in the low frequency band.

So far we have considered the stability of the ILC problem. Perhaps more important is the asymptotic performance of the ILC system, and the ability to achieve perfect tracking. We give necessary and sufficient conditions for perfect tracking when $\|Q - LP\| < 1$ as follows:

Theorem 2.3 (Perfect Tracking)**.** *Let $P : U \to Y$ be linear and bijective. Assume that the conditions of Theorem 2.2 are satisfied for the ILC system described by (2.1) with the update law (2.2). Then the converged error equals* 0 *for all $r, d \in Y$ if and only if $Q = I$.*

Proof. Given $r, d \in Y$, by Theorem 2.2, the system is monotonically convergent with the fixed point $\bar{u}$ satisfying $(I - Q)\bar{u} = L\bar{e}$; $\bar{e} \triangleq r - d - P\bar{u}$. Sufficiency follows directly from this relationship since: 1) The contraction condition, along with the fact that $Q = I$ means L is nonsingular; 2) Bijectivity of P implies that $\bar{u}$ uniquely satisfies $\bar{e} = 0$. For necessity, note that $(I - Q)\bar{u} = 0$ must hold for all $\bar{u} \in U$, thus $Q = I$. ■

Under the conditions of the perfect tracking theorem, the optimal learning operator L in terms of fastest convergence is obviously P^{-1}. Naturally, one can ask if ILC is necessary at all when P is known perfectly and is invertible. Indeed, such a statement would be valid when $d = 0$. For the nontrivial case of $d \neq 0$, one can achieve perfect tracking by running a single trial to identify d and correcting for it in the second trial by setting $u = P^{-1}(r - d)$. But this is no different than running the ILC system for two trials with $L = P^{-1}$! The ILC algorithm formalizes this procedure, with the additional benefit of achieving zero errors under uncertainties in P.

Of course, given more detailed information about a particular system or a problem setting it makes sense to incorporate more advanced methods to improve on desired properties such as robustness, convergence speed and asymptotic performance. We will look at a particular example of this situation in the next section.

2.2 Discrete Time and the Supervector Framework

Discrete time linear systems in ILC have a significant body of literature. This is due to the fact that such systems can easily be represented in matrix form, thus transforming the 2D ILC problem into the *feedback control problem* of a static multiple-input multiple-output (MIMO) system. Of course, more generally, this statement is true for any setting where the input and output spaces are finite dimensional; e.g. linear systems with spatial or spatiotemporal dynamics that can be discretized.

Consider the following SISO discrete LTI system

$$\begin{aligned} x_k(t+1) &= Ax_k(t) + Bu_k(t), \quad x_k(0) = x_0, \\ y_k(t) &= Cx_k(t), \end{aligned}$$

for all $t \in \{0, 1, \ldots, T\}$ and $k \in \mathbb{N}$, where $x_k(t) \in \mathbb{R}^n$ is the state vector, $u_k(t) \in \mathbb{R}$ is the input, $y_k(t)$ is the output, and A, B, C are appropriately sized real matrices. The solution of this system is given by

$$y_k(t) = (p * u_k)(t) + CA^t x_0, \quad \forall t \in \{0, 1, \ldots, T\}, \forall k \in \mathbb{N} \tag{2.3}$$

where $p(t)$ is the impulse response of the system with the Markov parameters given by $p(t) = CA^tB$ for all $t \in \{0, 1, \ldots, T\}$, and $*$ is the convolution operator. Assuming a relative degree or delay of 1, and letting $d(t) = CA^t x_0$, the system can be cast into the following matrix form:

$$\underbrace{\begin{bmatrix} y_k(1) \\ y_k(2) \\ \vdots \\ y_k(T) \end{bmatrix}}_{y_k} = \underbrace{\begin{bmatrix} p(0) & 0 & \dots & 0 \\ p(1) & p(0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p(T-1) & p(T-2) & \dots & p(0) \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} u_k(0) \\ y_k(1) \\ \vdots \\ y_k(T-1) \end{bmatrix}}_{u_k} + \underbrace{\begin{bmatrix} d(1) \\ d(2) \\ \vdots \\ d(T) \end{bmatrix}}_{d}, \tag{2.4}$$

for all $k \in \mathbb{N}$. The error vector $e_k(t) = r(k) - y_k(t)$ can be written in matrix form in a similar manner. This procedure is called *lifting* in the ILC literature, and the system (2.4) is said to be in *lifted form* with the *supervectors* y_k, u_k, d, e_k. As with the abstract representation of the previous section, the formulation $y_k = Pu_k + d_k$ is a sufficiently general form for iteration invariant discrete time linear systems, where the effects of initial conditions, noise, and disturbance can be collected in the vector d_k. Causality implies that P has to be lower triangular[10], and time invariance means that P is Toeplitz, as can be seen in (2.4). This formulation can be easily generalized to time varying and/or noncausal systems, and systems with relative degree higher than 1. The standard assumption is that no matter the structure or properties of P and d, they are *invariant over the iteration axis*. The general linear update law is then given by

$$u_{k+1} = Qu_k + Le_k, \quad \forall k \in \mathbb{N}, \tag{2.5}$$

with arbitrary u_0.

Since the matrices Q and L can be thought of as lifted forms of linear operators in $\mathbb{R}^T$, from here onwards, we will use the term time invariant to signify a Toeplitz matrix. Similarly, causality will signify a lower triangular matrix. With this terminology, we are ready to state the asymptotic stability condition for discrete LTI systems.

[10]Similarly, an *anticausal* operator has to be upper triangular.

Theorem 2.4 (Asymptotic Stability)**.** *The ILC system (2.4) with the update law (2.5) is asymptotically stable if and only if* $\rho(Q - LP) < 1$*. If* Q *and* L *are causal time invariant matrices, the condition simplifies to* $|q_0 - l_0 p_0| < 1$*, where* q_0 *and* l_0 *are the first nonzero Markov parameters of* Q *and* L*.*

Proof. The asymptotic stability condition follows directly from linear systems theory. When Q and L are causal and time invariant, $Q-LP$ is lower triangular and Toeplitz with $q_0 - l_0 p_0$ on the main diagonal. Hence, $q_0 - l_0 p_0$ is the only distinct eigenvalue of $Q - LP$, so the system is asymptotically stable if and only if $|q_0 - l_0 p_0| < 1$. ■

A couple of remarks are in order here: First, as asymptotic stability implies bounded-input bounded-output (BIBO) stability, u is bounded. Second, the simple, model free[11] nature of ILC can be observed with the asymptotic stability condition. The next theorem is the specialization of Theorem 2.2 for the discrete time case. As before, monotonic convergence here means the exponential convergence of the input vector.

Theorem 2.5. *The ILC system (2.4) with the update law (2.5) is monotonically convergent in a given vector norm if* $\|Q - LP\| < 1$*, where* $\|.\|$ *is the associated induced matrix norm.*

Corollary 2.2. *The ILC system (2.4) with the update law (2.5) is asymptotically stable if the norm condition of Theorem 2.5 is satisfied.*

Proof. Follows from Theorem 2.4 and the fact that the spectral radius is a lower bound on any induced matrix norm. ■

[11]The term model free is in reality a misnomer and should be used cautiously. In this context, it means that the knowledge of the first nonzero Markov parameter, or knowledge of the matrices B, C is necessary and sufficient to design a stable update law. No prior knowledge of A is necessary. This is similar to some traditional adaptive control schemes where only limited information about the system, such as its relative degree and the sign of high frequency gain, is assumed.

2.3 Higher Order Algorithms

The standard first order linear algorithm can be generalized to an nth order algorithm by the following formula:

$$u_{k+1} = -(Q_n u_k + Q_{n-1} u_{k-1} + \cdots + Q_0 u_{k-n}) + L_n e_k + L_{n-1} e_{k-1} + \cdots + L_0 e_{k-n}, \quad (2.6)$$

for all $k \in \mathbb{N}$ by taking $e_l = u_l = 0$ for $l < 0$. Here, Q_i and L_i can be thought of as linear operators in general, which would simplify to a matrix representation in finite dimensions. For this case, some authors [50] have defined the so called w transform, where w is a one step trial shift operator, e.g. $w^{-1}u_k(t) = u_{k-1}(t)$. The w transform is no different than the familiar z transform, renamed to emphasize the fact that it operates pointwise on the trial domain. With this, the higher order algorithm (2.6) can be written in the form

$$u(w) = C(w)e(w); \quad C(w) \triangleq Q_c^{-1}(w)L_c(w),$$

where

$$Q_c(w) \triangleq I w^{n+1} + Q_n w^n + \cdots + Q_1 w + Q_0,$$
$$L_c(w) \triangleq L_n w^n + L_{n-1} w^{n-1} + \cdots + L_1 w + L_0.$$

Again, we are reminded of the fact that lifting the discrete time system transforms the 2D ILC problem into a standard feedback control problem, where higher order algorithms can be designed using existing methods from linear multivariable feedback control. Checking the stability of the ILC system then reduces to checking the stability of the following transfer matrix:

$$G(w) \triangleq (Q_c(w) + L_c(w)P)^{-1} L_c(w)P = P(Q_c(w) + L_c(w)P)^{-1} L_c(w),$$

where P is the plant input-output matrix. For the special case of

$$u_{k+1} = (I - Q_{n-1})u_k + (Q_{n-1} - Q_{n-2})u_{k-1} + \cdots + (Q_1 - Q_0)u_{k-n+1} + Q_0 u_{k-n}$$
$$+ L_n e_k + L_{n-1}e_{k-1} + \cdots + L_1 e_{k-n+1} + L_0 e_{k-n},$$

we have

$$u(w) = \frac{1}{w-1} C(w)e(w),$$

so

$$G(w) \triangleq ((w-1)Q_c(w) + L_c(w)P)^{-1}L_c(w)P = P(Q_c(w) + L_c(w)P)^{-1}L_c(w). \quad (2.7)$$

Equation (2.7) reflects the integrator action of ILC on the iteration domain. By the final value theorem, if G is stable, $e_k \to 0$.

CHAPTER 3

Literature Review of ILC

In this chapter, we will do a short review of ILC literature. The chapter will be divided into two sections: In Section 3.1, we will cover publications on general ILC theory and some practical applications. Section 3.2 will focus on literature specific to some of the research questions.

3.1 A Short History of ILC

The works of Uchiyama [3] are the first publications to appear on ILC. The learning paradigm was arguably first formalized by Arimoto [4] on 1984, although independent rigorous treatments of the problem were developed simultaneously by Craig [5], and Casalino and Bartolini [6]. Interestingly, the central idea of learning from repetition has appeared in the literature as early as the 1970s [7, 51], and even before in a US patent filed in 1967 [52]. The two recent surveys by Bristow et al. [9] and Ahn et al. [12] are currently the most extensive resources on ILC. We also note three other surveys that have appeared in the 1990s [51, 53, 54]. In addition to these surveys, several monographs [7, 32, 55–59] and special issues [60–62] have appeared since then. A good starting point for the working engineer is [49], where the exposition is restricted to discrete LTI systems. Although, not exclusively on ILC, the edited volume Iterative Identification and Control [63] contains interesting ideas that link feedback, adaptation, identification, and ILC. Finally, recalling that ILC is a special class of repetitive processes, the 2007 monograph by Rogers et al. [2] can also be regarded as an important reference, wherein a systematic study of the stability, robustness, and optimality of linear repetitive processes is conducted.

Without going into extensive literature on the subject, we also note that repetitive control[12] is a similar area of research, with the main difference from ILC being that repetitive control is intended for continuous operation, whereas ILC has a discrete nature: In repetitive control, the objective is to improve the tracking performance of a system that is subject to periodic references or disturbances, e.g. a rotating hard disk drive head [9]. As opposed to ILC, this implies that the terminal condition of a period dictates the initial condition of the next period, leading to different analysis techniques and results [43, 64].

A comprehensive pool of references on ILC theory and applications, along with taxonomy and categorization of these works into subfields can be found in the most recent surveys [9, 12]. In the following subsections we will see the broad picture on the state of the art on ILC theory and applications.

3.1.1 Theoretical Works

ILC theory has a vast body of literature and includes, for example, feedback equivalence [18–22], higher order algorithms [65, 66], 2D systems based design and analysis [67, 68], among others. The three main subfields, as it relates to feedback control theories, can be stated as robust ILC, optimal ILC, and adaptive control based ILC.

3.1.1.1 Robustness in ILC

As in feedback control, robustness is a central issue in ILC. Works in this category deal with disturbance rejection [69], plant uncertainties [32, 70], stochastic noise [32, 71], using $\mathcal{H}_\infty$ methods [70, 72, 73], μ synthesis [74], interval uncertainties and model

[12] Not to be confused with repetitive processes.

conversion [32]. Robust methods are well studied in the literature, although the majority of the results assume the uncertain plant descriptions and/or signals to be iteration invariant save for some preliminary work [69, 71, 75][13].

3.1.1.2 Optimal ILC

Optimization based ILC design is an active research area, perhaps due to the fact that it allows a systematic design of learning filters that guarantee stable behavior and minimize certain performance criteria. Indeed, the classical ILC problem can be cast into an optimization framework by requiring that the output converges to $\arg\min_{u\in\mathcal{U}\|r-Pu\|}$ [7]. Other references on optimal ILC include [76–79]. Another advantage to the optimization formulation is flexibility, where additional performance metrics can be included in the cost function especially for systems that have a degree of redundancy [14–17].

3.1.1.3 Adaptive Approaches in ILC

Adaptive control based methods are quite popular in ILC and are directly related to adaptive feedback control concepts. These methods provide a useful way of designing ILC algorithms for nonlinear systems and often times are extended from adaptive feedback controllers [80]. Another advantage to adaptive ILC (AILC) is the ability to incorporate varying references [81, 82]. AILC relies on certainty equivalence as in adaptive feedback, except that in the iterative case estimations are performed in a discrete manner during each trial. This also provides the ability to correct transient tracking errors in adaptive feedback [82]. Some other examples of AILC can be seen in [83, 84].

[13]We exclude ILC schemes based on higher order internal models in this statement; these will be elaborated on later on in Section 3.2.

3.1.2 ILC Applications

Examples of ILC applications in the literature are abundant; in particular, ILC implementations are common in the following application areas:

1. Robotics: Robotics is the most active area for ILC applications. As a matter of fact, Arimoto's original paper [4] develops the ILC algorithm on the basis that it can be used for improved performance in robotic manipulators. ILC applications in robotics are numerous, for example see [79, 85–88].

2. Rotary Systems: Rotary systems are suitable candidates for ILC implementation due to the implicit spatial or temporal periodicity [89, 90].

3. Manufacturing and Batch Processes: Batch processes commonly use a combination of feedback (for example, model predictive control [91]) and ILC algorithms [92]. Among a number of manufacturing applications, semiconductor production widely uses ILC as a compensation tool [73, 93]. ILC is a very effective control strategy for manufacturing applications, especially in situations where online sensing is challenging or infeasible [94].

4. Bioengineering Applications: Biomedical and biomechanics applications is an emerging research field in ILC, see for instance [95–97].

5. Actuator Nonlinearity Compensation: ILC is used in systems to compensate for actuator nonlinearities (deadzone, hysteresis, backlash) [98] in precision motion control applications [99].

3.2 Relevant Literature for the Research Problems

We first discuss the literature related to the research directions pointed out in [31–33]. Of these directions, nonlinear update laws are the most active area of research in the literature. However, the analysis is constrained to AILC as pointed out in [33]. Spatial ILC is a recent area that has attracted attention [94, 100]. Performance analysis and guidelines for linear ILC is increasingly prevalent; for example refer to [32, 101, 102]. Finally, network controlled systems are briefly discussed in [32].

The research questions we posed in Chapter 1 can be roughly condensed to the single question of whether the fundamental iteration invariance assumption in ILC, which was discussed in Section 1.1, can be relaxed. To date, there has been limited material that has attempted to relax these assumptions. Among these, initial condition invariance is by far the most discussed topic since perfect resetting can be hard to achieve for certain systems [103]. The central result in [103] is that the varying system (that is, the system subject to an initial condition resetting error) converges to a bounded neighborhood of the invariant system when the resetting error is uniformly bounded. Varying references are also increasingly studied in ILC theory; AILC is one of the avenues in which this objective is pursued [81], while some other works consider parametrizing the set of references by basis functions [34, 104, 105] or library based interpolations [35]. Varying disturbance signals have been studied in stochastic settings [32, 69, 71]. Ref. [106] considers varying time intervals through the use of a time scale transformation. Lastly, iteration varying plant models are actively studied in the case that they can be described by a higher order internal model (HOIM) [107]; that is systems wherein the plant operator P_k at trial k is a function of $P_{k-1}, P_{k-2}, \ldots, P_{k-n}$ for some n, although to the best of our knowledge, there has been no studies on whether HOIMs occur naturally in physical systems.

CHAPTER 4

Robust ILC through $\mathcal{L}_1$ Adaptive Feedback

In this chapter, we tackle the robust monotonic convergence problem of uncertain linear systems for high precision tracking performance. The problem will be discussed for continuous time systems under parametric uncertainties. A practical motivation for this study comes from precision motion control applications, where demanding design specifications pose a large array of control challenges. As a result, precision motion control design relies on a variety of advanced control strategies developed to cope with specific problems present in control theory. Although ILC can decrease tracking errors up to several orders of magnitude for repetitive tasks, the achievable performance is limited by dynamic uncertainty. Thus, in this chapter, we propose the combination of $\mathcal{L}_1$ adaptive control ($\mathcal{L}_1$ AC) and linear ILC for precision motion control under parametric uncertainties. We will rely on the adaptive loop to compensate for parametric uncertainties, and ensure that the plant uncertainty is sufficiently small so that an aggressive learning controller can be designed on the nominal system. We will exploit the closed loop stability condition of $\mathcal{L}_1$ AC to design simple, robust ILC update laws that reduce tracking errors to measurement noise for time varying references and uncertainties. Finally, we will demonstrate in simulation that the combined control scheme maintains a highly predictable, monotonic system behavior; and achieves near perfect tracking within a few trials regardless of the level of uncertainty in the system.

Of course, combining ILC with feedback control techniques to achieve high performance is not a novel idea by any means[14]. This is because essentially, the achievable performance through learning is limited by the closed loop dynamics, much as a human is constrained by the dynamic limitations of his or her neuromuscular system when learning to perform a motion task[15]. Theoretically, our approach is motivated by an effort to relax the plant invariance assumption, even though the analysis will assume that the uncertainties are invariant from iteration to iteration. This is similar to conventional adaptive control, wherein the objective is to adapt to changing conditions to provide high tracking performance under uncertainty, although most of the basic theory assumes time invariant uncertainties[16]. In that sense, adaptive control is a good feedback strategy to explore the effects of iteration varying uncertainties. $\mathcal{L}_1$ AC, in particular, has certain benefits over conventional adaptive control, such as arbitrarily close uniform tracking of a linear reference model, which makes it a suitable feedback control candidate from a learning perspective. The simulation example presented at the end of the chapter shows the importance of considering iteration varying effects in a more explicit manner. A more detailed motivation for our approach and a summary of the technical material is given in the following section.

4.1 On Robust ILC and $\mathcal{L}_1$ Adaptive Control

Recall from Chapters 1 and 2 that dynamic uncertainty is an essential challenge motivating the field of ILC. Much as in feedback control, the main approaches for

[14] Some authors even consider feedback control directly in an ILC framework, where the update law is modified to include a feedback term; often called "current cycle feedback" or " current iteration feedback" [7, 9].

[15] It can be argued that this is not the case when perfect tracking can be achieved. However, even when such an objective is theoretically feasible, it is rarely achieved in practice. For example, stochastic measurement noise is a major obstacle to the tracking objective, and the statistical characteristics of this random process can be amplified by feedback.

[16] Another way of interpreting this is that the time scale of the parameter variations is much slower than the time scales of the estimation and control loops [108].

mitigating uncertainties in ILC can be roughly classified as robust or adaptive methods. As we have discussed in Chapter 3, there has been a significant body of research on both robust and adaptive ILC methodologies. The main drawback of robust ILC methods is that while ILC convergence is guaranteed within the prescribed set of uncertainties, performance is often limited due to conservative designs. Additionally, the sensitivity of robust learning controllers to variations in the uncertainties is still an open question. On the other hand, while the adaptive nature of AILC schemes signify high performance and reduced sensitivity to parametric variations, the robustness of adaptive ILC to unmodeled dynamics may be questionable, analogous to adaptive feedback control [109, 110].

Most of the fundamental limitations and trade-offs of control theory can be observed to a greater extent in precision motion control due to complex, demanding design specifications. Key issues in the control of precision positioning systems include robustness to parameter variations, unmodeled high frequency dynamics, and the bandwidth-precision trade-off [99]. More complex process modeling can mitigate uncertainty issues to an extent, but this becomes unfeasible as complexity increases, specifically due to the fact that certain information about the process, such as external loads and/or parameters that are sensitive to exogenous effects, cannot be known a priori. Although adaptive feedback methods provide a good solution to the problem of *robustness to parametric variation* and increase precision, this often comes at the expense of *reduced robustness to unmodeled dynamics* [110] as fast estimation, which is desired from a performance standpoint, leads to high gain feedback. This problem essentially boils down to the fact that conventional adaptive control ignores Bode's sensitivity integral [111, 112], also known as the waterbed effect, by compensating for uncertainties throughout the whole frequency spectrum. Similarly, while ILC extends the *available bandwidth* [112] of the control channel for repetitive systems, thereby alleviating the bandwidth-precision trade-off, the achievable reduction in errors and monotonicity on the iteration axis depends largely on the level of uncertainty in the feedback stabilized plant.

Hence, to address these issues, we propose the combination of conventional ILC with $\mathcal{L}_1$ AC, a recent model reference adaptive control (MRAC) paradigm that bridges the gap between adaptive and robust control with a priori known, quantifiable transient response *and* robustness bounds [110]. The idea of combining ILC with $\mathcal{L}_1$ AC was first introduced in [113], wherein the adaptive loop was utilized to keep the plant sensitivity close to its nominal value for performance improvement through learning. Despite the displayed advantages of $\mathcal{L}_1$ AC over linear feedback, a trade-off was observed between the closed loop bandwidth and learning performance. More precisely, it was seen that higher closed loop bandwidths resulted in slower convergence and larger converged errors in the iteration domain. To resolve this problem, we proposed the augmentation of the $\mathcal{L}_1$ AC architecture with an arbitrary feedforward signal to accommodate learning, leading to an adaptation that considers changes in the nominal system trajectories due to learning [36]. The resulting $\mathcal{L}_1$ AC-ILC ($\mathcal{L}_1$-ILC) scheme had predictable performance in both the time and iteration domains: The feedforward augmented closed loop preserved the a priori known quantifiable transients from $\mathcal{L}_1$ AC theory, and the learning controller displayed similar convergence behavior regardless of the uncertainty present in the system. It was also seen that increasing feedback bandwidths resulted in decreasing effects of uncertainty in the iteration domain, with faster convergence and lower converged errors. In [37], we presented design guidelines and showed the performance gains of the modified scheme over linear output feedback on a large range nanopositioner via simulation.

The material that we discuss in this chapter was originally presented in [38] and is a generalization of $\mathcal{L}_1$-ILC to different classes of systems through vector space methods. We demonstrate how ILC algorithms can be combined with $\mathcal{L}_1$ AC schemes to achieve robust, high precision motion control. We present feedforward augmented $\mathcal{L}_1$ AC architectures for state and output feedback cases (see Figures 4.2 and 4.5) to accommodate parallel ILC signals and show how this preserves the a priori known $\mathcal{L}_1$ AC transient bounds. We explain how these bounds, which imply arbitrary close tracking of *linear* reference models in the time domain, can be exploited for learning purposes in the iteration domain. We then show how the $\mathcal{L}_1$ AC stability condition relates directly to the robust monotonic convergence conditions of LTI learning laws, and how robust ILC algorithms can be designed in a simple, straightforward manner for different $\mathcal{L}_1$ AC architectures.

The rest of the material in this chapter is organized as follows. Section 4.2 introduces some preliminaries for clarity of exposition. Section 4.3 gives a brief introduction to $\mathcal{L}_1$ AC and ILC, and presents our proposed method for the state feedback case. Section 4.4 extends the results to time varying uncertainties in output feedback. Simulation results are given in Section 4.5. Section 4.6 gives concluding remarks and summarizes our findings. For a streamlined presentation, we give certain intermediate results in Appendix A.1, proofs of our main results in Appendix A.2 and several auxiliary variables in Appendix A.3.

4.2 Notation and Preliminaries

Throughout this chapter, we use time and frequency domain representations interchangeably for signals. For example, $f(s)$ denotes the Laplace transform of the signal $f(t)$. We denote systems and matrices with upper case letters. We represent signals and vectors with lower case letters. We use script letters to distinguish linear operators in general from their matrix and transform representations (e.g. $\mathcal{F}$ instead of $F(s)$). We take $\mathbb{R}$ to represent the set of real numbers and $\mathbb{R}^+$ the set of positive real numbers. We choose $\mathbb{C}$ to denote complex numbers. We take I to be the identity matrix of appropriate size and $\mathcal{I}$ to be the identity operator in the relevant space.

We use $\lambda_{\max}(.)$ and $\lambda_{\min}(.)$ to denote the maximum and minimum eigenvalues of a positive definite matrix, respectively. We take $\|.\|_p$ for $p \in [1, \infty]$ as the standard vector and induced p norm. We use F^T for the transpose of a matrix F.

In the rest of the section, we collect several definitions and facts from systems theory pertinent to our discussion.

Definition 4.1. For any $p \in [1, \infty)$, $\mathcal{L}_p^n$ is defined as the space of all piecewise continuous $f : \mathbb{R} \to \mathbb{R}^n$ such that $\|f\|_{\mathcal{L}_p} \triangleq (\int_{-\infty}^{\infty} \|f(t)\|^p dt)^{1/p} < \infty$, where $\|.\|$ is any standard vector norm in $\mathbb{R}^n$. However, it is conventional to use the 2 norm for $\mathcal{L}_2^n$. Similarly, $\mathcal{L}_\infty^n$ is defined as the space of all piecewise continuous $f : \mathbb{R} \to \mathbb{R}^n$ such that $\|f\|_{\mathcal{L}_\infty} \triangleq \sup_{t \in \mathbb{R}} \|f(t)\|_\infty < \infty$.

Definition 4.2. For any $p \in [1, \infty]$, the extended space $\mathcal{L}_{pe}^n$ is defined as the space of all piecewise continuous causal $f : \mathbb{R} \to \mathbb{R}^n$ such that $\|f_\tau\|_{\mathcal{L}_p} < \infty \; \forall \tau \geq 0$, where f_τ is the truncation of f defined by $f_\tau(t) \triangleq f(t)$ for $0 \leq t \leq \tau$ and $f_\tau(t) \triangleq 0$ for $t > \tau$.

Definition 4.3. For a given m input n output LTI system $F(s)$ with impulse response $f(t) \in \mathbb{R}^{n \times m}$, the $\mathcal{L}_1$ norm is defined as $\|F(s)\|_{\mathcal{L}_1} \triangleq \max_{k \in 1,2,\ldots,n} \sum_{l=1}^{m} \|f_{kl}\|_{\mathcal{L}_1}$, where $f_{kl}(t)$ is the entry at the kth row and lth column of $f(t)$.

Definition 4.4. The $\mathcal{L}_\infty(j\mathbb{R})$ norm of a BIBO stable LTI system $F(s)$ is defined by $\|F(s)\|_\infty \triangleq \sup_{\omega \in \mathbb{R}} \|F(j\omega)\|_2$.[17]

Lemma 4.1. *Let $F(s)$ be a stable causal LTI system. Then for every bounded input ζ, the output ξ is bounded and we have $\|\xi_\tau\|_{\mathcal{L}_\infty} \leq \|F(s)\|_{\mathcal{L}_1} \|\zeta_\tau\|_{\mathcal{L}_\infty}$ [110, page 273].*

Remark 4.1. Lemma 4.1 shows that the $\mathcal{L}_1$ norm of a stable LTI system is essentially its *induced $\mathcal{L}_\infty$ norm*: If $\|F(s)\|_{\mathcal{L}_1} = \sum_{l=1}^{m} \|f_{kl}\|_{\mathcal{L}_1}$ for some k, the equality can be achieved by taking $u_l(t - \upsilon) = \text{sgn}(f_{kl}(\upsilon))$. Consequently, an LTI system $F(s)$ is

[17]The $\mathcal{L}_\infty(j\mathbb{R})$ norm of a transfer function should not be confused with the $\mathcal{L}_\infty$ norm of a signal in the time domain. For causal $F(s)$, $\|F(s)\|_\infty$ is precisely the $\mathcal{H}_\infty$ norm.

BIBO stable if and only if $\|F(s)\|_{\mathcal{L}_1} < \infty$ [110, page 274], which justifies the use of the $\mathcal{L}_1$ norm in establishing boundedness in $\mathcal{L}_1$ AC algorithms.

Theorem 4.1. *For a BIBO stable LTI system $F(s)$ the induced $\mathcal{L}_2$ norm is equal to $\|F(s)\|_\infty$ [11, page 101].*

Readers will note that we mainly consider two types of signal norms: $\mathcal{L}_\infty$ and $\mathcal{L}_2$. The $\mathcal{L}_\infty$ norm will be used in $\mathcal{L}_1$ AC to establish boundedness (Lemma 4.1), while the $\mathcal{L}_2$ norm will be of interest in ILC as a performance metric. The following will be used in establishing the relationship between the two for ILC design:

Lemma 4.2. *For a stable causal n output LTI system $F(s)$, $\|F(s)\|_\infty \leq \sqrt{n}\|F(s)\|_{\mathcal{L}_1}$.*

Proof. See Appendix A.2. ■

Remark 4.2. While Lemmas 4.1 and 4.2 are given for causal systems, the results are also true in essence for noncausal systems. For example, for a stable noncausal LTI system $F(s)$ with bounded input ζ, $\|\xi\|_{\mathcal{L}_\infty} \leq \|F(s)\|_{\mathcal{L}_1}\|\zeta\|_{\mathcal{L}_\infty}$, where ξ is the output. In the rest of the chapter, unless otherwise noted, we will assume all systems and signals to be causal.

4.3 State Feedback

We will start our discussion with the full state feedback $\mathcal{L}_1$ AC architecture (Figure 4.1) for SISO LTI systems with unknown pole locations. This class of systems offers a good introduction to $\mathcal{L}_1$ AC and will show us that the guaranteed transient property holds with the addition of a feedforward signal *in the problem objective.* We will then demonstrate how this property, along with the main stability condition of $\mathcal{L}_1$ AC, can aid us in the design of our learning law. Finally, we will have a brief look at the design trade-offs and argue how $\mathcal{L}_1$ AC and ILC can be combined into a

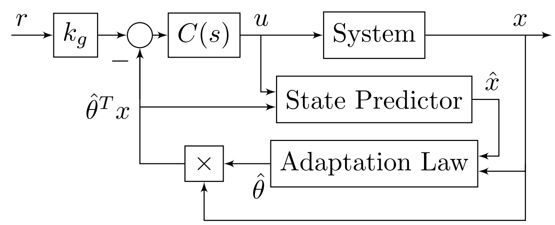


Figure 4.1: $\mathcal{L}_1$ adaptive control for LTI state feedback with unknown pole locations

single framework with the unified objectives of high tracking performance, robustness to uncertainties and monotonic transient response in the time and iteration domains.

4.3.1 $\mathcal{L}_1$ Adaptive Control

$\mathcal{L}_1$ AC is a recently developed model following control methodology [110] with guaranteed transient performance and robustness in the presence of fast adaptation. The central idea of $\mathcal{L}_1$ AC theory lies in the use of the available bandwidth of the control channel, imposed by physical hardware [112]. Drawing inspiration from robust and classical control, $\mathcal{L}_1$ AC aims to compensate for uncertainties in a limited range of frequencies, a more "feasible" objective than that of conventional MRAC wherein uncertainties are compensated over the whole spectrum. This approach brings significant advantages over conventional MRAC, the most critical of these being the "decoupling" of estimation and control, realized by the presence of a bandlimited filtering structure at a particular point[18] in the architecture. As a result of this property, the performance-robustness trade-off of $\mathcal{L}_1$ systems is defined by the bandwidth of the filter as opposed to the rate of adaptation. This trade-off can be addressed with tools from classical and robust control; whereas the adaptation rates can be increased arbitrarily and are limited only by practical concerns such as hardware speed and noise. Consequently, uniform performance bounds on the input and output signals can be enforced by high adaptation rates while still maintaining a relatively high level

[18]This particular point varies depending on the class of systems, see for instance Figure 4.1 for the LTI state feedback with unknown pole locations.

of robustness [114].

In spite of the advantages we have laid out, there have been a number of publications to date that have questioned the merits of $\mathcal{L}_1$ AC, claiming that 1) $\mathcal{L}_1$ AC offers "no benefits in terms of robustness, performance, or bounds that suggest useful trade-offs", 2) the reference model stability condition cannot be satisfied for certain plants and reference models, and 3) adaptation is unnecessary in $\mathcal{L}_1$ AC in some sense. Claim 3) has important implications from a robust control point of view, and shows that for certain classes of $\mathcal{L}_1$ controllers, there exists an equivalent implementable nonadaptive control law. However, for the architectures we consider, this requires the relative degree of the filtering structure to be higher than that of the plant. This issue is discussed in detail in Appendix A.4, along with other issues raised in the literature.

$\mathcal{L}_1$ AC algorithms have been developed for a wide range of classes. In this section, we present the $\mathcal{L}_1$ architecture for SISO LTI systems with unknown constant parameters. To account for changes in system trajectory due to feedforward control, and put the problem into a meaningful format, we augment the original controller [110] with a bounded feedforward signal.

4.3.1.1 Problem Formulation

We consider the following class of systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b(u(t) + \theta^T x(t)), \quad x(0) = x_{\text{in}}, \\ y(t) &= c^T x(t), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the measured state vector; $u(t) \in \mathbb{R}$ is the input; $b, c \in \mathbb{R}^n$ are known constant vectors; $A \in \mathbb{R}^{n \times n}$ is a known constant matrix, with (A, b) controllable; $\theta \in \Theta$ is an unknown constant vector contained in the compact convex set Θ; and $y(t) \in \mathbb{R}$ is the output signal. Without loss of generality, let A be Hurwitz.

Assumption 4.1. The set $\Theta = \{\theta \in \mathbb{R}^n : \|\theta\|_\infty \leq \theta_{M_\infty}\}$ for some $\theta_{M_\infty} \in \mathbb{R}^+$.

Remark 4.3. Assumption 4.1 will enable us to abuse the relationship of Lemma 4.2 for ILC purposes.

The $\mathcal{L}_1$ AC objective is to track a given reference system in transient and steady state phases.

4.3.1.2 Closed Loop Reference System

The reference system dynamics are described by $(A, b, c^T, 0)$, the strictly proper BIBO stable transfer function $C(s)$ with DC gain 1 and zero state space initialization, and the unknown parameter θ. $C(s)$ is also subject to the $\mathcal{L}_1$ norm condition

$$\|G(s)\|_{\mathcal{L}_1}\theta_{M_1} < 1, \tag{4.2}$$

where $G(s) \triangleq H_x(s)(1-C(s))$, $H_x(s) \triangleq (sI-A)^{-1}b$; and $\theta_{M_1} \triangleq \max_{\theta\in\Theta} \|\theta\|_1 = n\theta_{M_\infty}$. Let $H(s) \triangleq c^T H_x(s)$. The feedforward augmented closed loop reference system can be defined as

$$\begin{aligned} \dot{x}_{\text{ref}}(t) &= Ax_{ref}(t) + b(u_{\text{ref}}(t) + \theta^T x_{\text{ref}}(t)), \quad x_{ref}(0) = x_{in}, \\ y_{\text{ref}}(t) &= c^T x_{\text{ref}}(t), \\ u_{\text{ref}}(s) &= C(s)(k_g r(s) - \theta^T x_{\text{ref}}(s)) + u_i(s), \end{aligned} \tag{4.3}$$

where $k_g = 1/H(0)$ is a static precompensator; $r(s)$ is the reference signal; and $u_i(s)$ is a bounded input signal in Laplace notation.

By augmenting the reference system with a feedforward control signal, we reformulate the problem so that the objective is to track certain given dynamics driven by a reference signal *and* a feedforward signal. Since this signal will be synthesized by certain filtering methods, and be used later for performance improvement, we choose not to pass it through $C(s)$. Note that letting $C(s) = 1$ in (4.3) results in the nominal system given by $(A, b, c^T, 0)$ (i.e. $\theta = 0$) with input $k_g r(t) + u_i(t)$. In that sense, we aim to only partially compensate for uncertainties, within the bandwidth of $C(s)$.

Lemma 4.3. *If (4.2) is satisfied, the reference system (4.3) is bounded-input bounded-state bounded-input bounded-state (BIBS) stable.*

Proof. See [110]. The proof follows in the same manner from the boundedness of $u_i(t)$. ■

Remark 4.4. Condition (4.2) ensures that the feedback gain of θ on the system states is small enough for stability (see Lemma A.1 in Section A.1) since $\|\theta\|_1$ is the $\mathcal{L}_1$ norm of the static LTI system θ^T. In other words, we require the bandwidth of $C(s)$ be high enough for sufficient compensation of uncertainties.

4.3.1.3 $\mathcal{L}_1$ Adaptive Controller

The $\mathcal{L}_1$ adaptive controller is based on a fast estimation scheme which consists of a state predictor, the bounded feedforward input $u_i(t)$ and the bandlimited filter $C(s)$.

State Predictor The controller relies on the following state predictor

$$\dot{\hat{x}}(t) = A\hat{x}(t) + b(\hat{\theta}^T(t)x(t) + u(t)) - K_{\mathrm{sp}}\tilde{x}(t), \quad \hat{x}(0) = x_{in}, \tag{4.4}$$

where $\hat{x}(t)$ is the state prediction vector; $\hat{\theta}(t)$ is the estimate of the unknown constant vector θ; $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$ is the prediction error; and $K_{\mathrm{sp}} \in \mathbb{R}^{n\times n}$ can be used to assign faster poles to $(A - K_{\mathrm{sp}})$ [115].

Adaptation Law The adaptation law that estimates θ is

$$\dot{\hat{\theta}}(t) = \Gamma\,\mathrm{Proj}(\hat{\theta}(t), -\tilde{x}^T(t)Pbx(t)), \quad \hat{\theta}(0) = \hat{\theta}_{in}, \tag{4.5}$$

where the arbitrary initial condition $\hat{\theta}_{in} \in \Theta$, $\mathrm{Proj}(.,.)$ is the projection operator defined in [116], with projection bound $\theta_{M_2} \triangleq \max_{\theta\in\Theta} \|\theta\|_2 = \sqrt{n}\theta_{M_\infty}$; $\Gamma > 0$ is

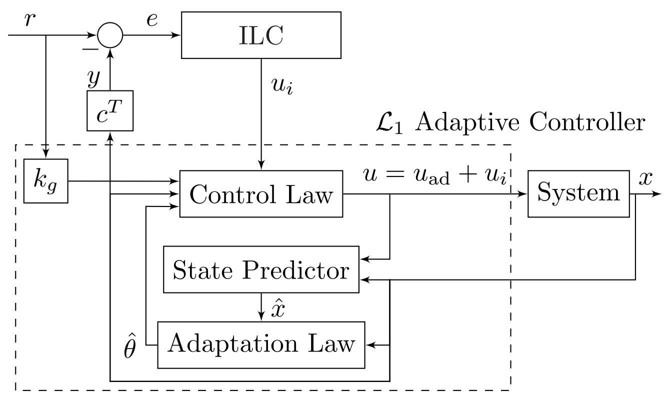


Figure 4.2: ILC with feedforward augmented $\mathcal{L}_1$ adaptive feedback

the adaptation rate; and $P = P^T > 0$ is the solution to the algebraic Lyapunov equation $A^TP + PA = -Z$, with arbitrary $Z = Z^T > 0$. The projection operator ensures the boundedness of $\hat{\theta}(t)$ by definition. This property is used extensively in the analysis of $\mathcal{L}_1$ schemes.

Control Law The control input is defined as

$$\begin{aligned} u(t) &= u_{\rm ad}(t) + u_i(t), \\ u_{\rm ad}(s) &\triangleq C(s)(k_g r(s) - \hat{\eta}(s)), \end{aligned} \tag{4.6}$$

where $u_{ad}(t)$ and $u_i(t)$ are the feedback and feedforward signals, respectively; and $\hat{\eta}(s)$ is the Laplace transform of $\hat{\theta}^T(t)x(t)$. Inclusion of the feedforward signal in the control input leads to the augmentation of the state predictor (see Figure 4.2). Hence, the controller generates the proper adaptive signal $u_{ad}(t)$ to track (4.3).

4.3.1.4 Transient Performance

The controller ensures transient and steady-state behavior in the input and output channels with respect to the $\mathcal{L}_1$ reference system, as stated in the theorem below.

Theorem 4.2. *For system (4.1) with the controller defined according to (4.4), (4.5), and (4.6), subject to the $\mathcal{L}_1$ norm condition (4.2); and its corresponding reference* *system (4.3), we have*

$$\begin{aligned} \|x_{\text{ref}} - x\|_{\mathcal{L}_\infty} &\leq \tfrac{\chi_1}{\sqrt{\Gamma}}, \quad \lim_{t\to\infty}(x_{\text{ref}}(t) - x(t)) = 0, \\ \|u_{\text{ref}} - u\|_{\mathcal{L}_\infty} &\leq \tfrac{\chi_2}{\sqrt{\Gamma}}, \quad \lim_{t\to\infty}(u_{\text{ref}}(t) - u(t)) = 0, \end{aligned} \tag{4.7}$$

where $\chi_1, \chi_2 \in \mathbb{R}$ *are defined in [110].*

Proof. See [110]. The proof is the same since $x_{\text{ref}}(t) - x(t)$ and $u_{\text{ref}}(t) - u(t)$ do not change with the choice of $u_i(t)$. ■

Theorem 4.2 implies that while itself being nonlinear, the $\mathcal{L}_1$ adaptive controller can track the linear reference model arbitrarily closely as Γ is increased. Since ILC uses information from the input and output channels, this property enables the use of the reference model in designing the ILC update law. Moreover, the reference system can be made arbitrarily close to the *design system*, at the expense of reduced robustness, by increasing the bandwidth of $C(s)$. For further details, we refer the readers to [110].

4.3.2 Iterative Learning Control

ILC architectures can be broadly classified as parallel or series in terms of their relation to feedback control loops. The parallel architecture, which we use in our controller (compare the $\mathcal{L}_1$ AC formulation in Section 4.3.1.2 with Figure 4.3), divides the input signal into feedback and feedforward components. In this approach, the feedforward signal for the next iteration is synthesized by processing the error and the feedforward input at the current iteration.

ILC design methods are numerous and include frequency domain, plant inversion, and optimization techniques. While frequency domain methods only approximate the system due to finite trial duration, they offer simplicity, flexibility and tunability as in classical control. For these reasons, we will be adopting frequency domain methods

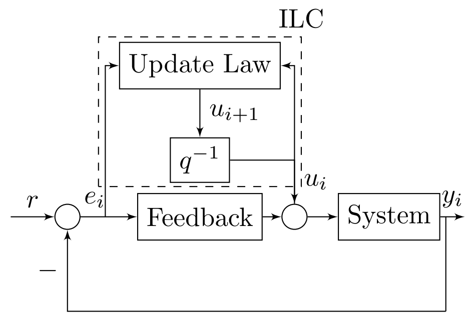


Figure 4.3: Parallel ILC Scheme with the forward trial shift operator q and update law $u_{i+1} = qu_i = f(u_i, e_i)$

to design our learning law.

4.3.2.1 Update Law

A common first order frequency domain ILC algorithm, which we will employ, is the Q filter and learning function approach[19]:

$$u_{i+1}(s) = Q(s)(u_i(s) + L(s)e_i(s)). \tag{4.8}$$

In (4.8), $u_i(s)$ is the ILC input; $Q(s)$ is the Q filter; $L(s)$ is the learning function; $e_i(s)$ is the reference tracking error; and i is the iteration index. In this approach, $L(s)$ is used to maximize learning, while $Q(s)$ limits the bandwidth for robustness and other practical purposes at the expense of performance. Asymptotic stability and monotonic convergence of the algorithm is given by the following well known theorem, which is a specific case of Theorem 2.2:

Theorem 4.3. *The ILC system, defined by the update law (4.8) acting on a stable SISO LTI system $F(s)$, is monotonically convergent if*

$$\|Q(s)(I - L(s)F(s))\|_\infty \leq \mu_F < 1$$

[19]This algorithm was discussed previously in Chapter 2 in general form; here we specialize it to the frequency domain.

for some μ_F. That is, $\|u_\infty - u_{i+1}\|_{\mathcal{L}_2} \leq \mu_F \|u_\infty - u_i\|_{\mathcal{L}_2}$, $i = 0, 1, \ldots,$ where $u_\infty(t)$ is the converged input.

Remark 4.5. As causality is not a constraint in ILC, the readers might ask if the condition is valid for noncausal $Q(s)$ and $L(s)$. The answer is yes, since the theorem is proven by defining a contraction in the input space $\mathcal{L}_2$ by aid of Theorem 4.1. Readers interested in the use of noncausal LTI operators in ILC can refer to [21].

4.3.2.2 Monotonic Convergence and Robustness

Recall the guaranteed transient property of the adaptive system as stated in (4.7). For the design of the update law, we will assume Γ is sufficiently high, and consequently that $x(t) = x_{\text{ref}}(t)$. Nevertheless, since the $\mathcal{L}_1$ controller aims to compensate for uncertainties within the bandwidth of $C(s)$, parametric uncertainties will still exist. The closed loop system can be described as

$$
\begin{aligned}
y_i(s) &= \bar{H}(s)u_i(s) + \bar{H}(s)C(s)k_g r(s) + c^T(I - G(s)\theta^T)^{-1}x_{\text{nr}}(s),\\
\bar{H}(s) &\triangleq c^T(I - G(s)\theta^T)^{-1}H_x(s) = \frac{H(s)}{1 - \theta^T G(s)},\\
x_{\text{nr}}(s) &\triangleq (sI - A)^{-1}x_{\text{in}},
\end{aligned}
$$

where the identity for $\bar{H}(s)$ follows from Lemma A.2.

While Theorem 4.3 ensures monotonic convergence in the $\mathcal{L}_2$ space for a nominal system, it does not guarantee the same under uncertainty. We now state our main result which shows that for the $\mathcal{L}_1$-ILC scheme, robust monotonic convergence can be guaranteed in a very simple way.

Theorem 4.4. *The ILC system with the update law (4.8) defined over $\bar{H}(s)$ subject to (4.2), is monotonically convergent with rate $\mu \in [0, 1)$ for all $\theta \in \Theta$ if*

$$
\kappa \leq \frac{\mu - |Q(j\omega)||1 - L(j\omega)H(j\omega)|}{|Q(j\omega)||L(j\omega)||H(j\omega)|}, \tag{4.9}
$$

for all $\omega \in \mathbb{R}$, where

$$\kappa \triangleq \frac{\theta_{M_2}\|G(s)\|_\infty}{1-\theta_{M_2}\|G(s)\|_\infty}. \tag{4.10}$$

Proof. Theorem 4.4 is a natural result of Theorem 6 of [9] when the uncompensated uncertainty is written in multiplicative uncertainty form. See Appendix A.2 for the details. ∎

Theorem 4.3 states that the nominal system can be rendered monotonically convergent by defining a contraction mapping in the input space. Theorem 4.4, on the other hand, directly extends monotonic convergence to the $\mathcal{L}_1$-AC scheme by making sure that the update law defines a contraction for all $\theta \in \Theta$. More specifically, (4.9) implies $\max_{\theta\in\Theta} \|Q(s)(1-L(s)\bar{H}(s))\|_\infty \leq \bar{\mu} < 1$ for some $\bar{\mu}$. This condition follows elegantly from the $\mathcal{L}_1$ norm condition which ensures that the plant uncertainty $(1-\theta^T G(s))^{-1}$ exists and is BIBO stable.

4.3.3 Design Trade-Offs

For a better understanding of the combined $\mathcal{L}_1$-ILC scheme, we will have a look at the design trade-offs. We first define $\Lambda(s) \triangleq (1-\theta^T G(s))^{-1}$. The inequalities below follow directly from the definitions of $\Lambda(s)$ and $\bar{\mu}$:

$$|\Lambda(j\omega)| \geq \frac{1}{|1-\theta^T H_x(j\omega)| + |C(j\omega)||\theta^T H_x(j\omega)|}, \tag{4.11}$$

$$\frac{\bar{\mu}}{|Q(j\omega)|} \geq |1-|L(j\omega)H(j\omega)\Lambda(j\omega)||. \tag{4.12}$$

It follows that

$$|L(j\omega)||H(j\omega)| \leq \left(\frac{\bar{\mu}}{|Q(j\omega)|}+1\right)\left(|1-\theta^T H_x(j\omega)| + |C(j\omega)||\theta^T H_x(j\omega)|\right). \tag{4.13}$$

Recall that $C(s)$ and $Q(s)$ describe the performance-robustness trade-offs in their respective domains. Thus, generally speaking, we can conclude the following:

1. Increasing the bandwidth of $C(s)$ decreases the minimum $\bar{\mu}$ that satisfies (4.13), i.e. faster convergence. Indirectly, a higher bandwidth also results in better iteration domain robustness since $\bar{\mu}$ becomes bounded further away from 1, thereby leaving the possibility of higher gain Q filters for enhanced performance: As the bandwidth of $C(s)$ increases, κ, as defined in (4.10), decreases since by definition $\|G(s)\|_\infty \leq \|H_x(s)\|_\infty \|1 - C(s)\|_\infty$. As a result, the designer can tune $Q(s)$ to increase its bandwidth and minimize the converged error.

2. Decreasing the bandwidth of $Q(s)$ decreases the minimum $\bar{\mu}$ that would satisfy (4.13), which signifies increased iteration domain robustness. This further implies that one can use a lower gain $C(s)$ for a feedback system with better stability margins: Because $Q(s)$ has a lower gain, there exists a higher value of κ satisfying (4.9) for the initial value of $\bar{\mu}$.

It thus makes sense to summarize the design trade-offs for the combined $\mathcal{L}_1$-ILC scheme as that of performance versus robustness. Intuitively, this is to be expected as increasing the passband of $C(s)$ decreases the uncertainty

$$\Lambda(s) = (1 - \theta^T H(s)(1 - C(s)))^{-1},$$

which is the desired result from an ILC perspective. For further insight into the controller, we refer the readers to [36] where we provide extensive simulations showcasing decreasing effects of uncertainty with increasing feedback bandwidth, and similar performance for all uncertainties and bandwidths such that the closed loop system remains stable.

4.3.4 Practical Considerations and Design Guidelines

The level of detail surrounding the previous sections may leave the impression that the design of the combined $\mathcal{L}_1$-ILC algorithm is highly complicated. In reality, despite the algebraic intensity of the analysis, the adaptive and learning controllers rely on fundamental ideas of classical and robust control. Hence, in this section we will explore how the controller can be designed in a relatively straightforward and systematic way using these ideas. The trade-offs given in Section 4.3.3 will be helpful towards that end.

The obvious starting point of this procedure is the design of the $\mathcal{L}_1$ adaptive feedback controller. Readers would note that the main design decisions of $\mathcal{L}_1$ AC are the bandwidth of the feedback filter $C(s)$ and the magnitude of the adaptation rate Γ. At this point we would like to direct the readers' attention to Theorem 4.2 and remind that the theoretical model tracking error of the feedback system can be set arbitrarily low. Therefore, the design of the filter and selection of the adaptation rate are decoupled. As we have mentioned in Section 4.3.3, $C(s)$ describes the performance-robustness trade-off in the time domain; i.e. a higher closed loop bandwidth results in decreased robustness margins and vice versa. Thus, the natural question that follows is if the $\mathcal{L}_1$ norm condition can be satisfied. The lemma below illustrates how this is indeed always possible:

Lemma 4.4. *Let* $F(s) = \frac{\prod_{k=1}^{m}(s+z_k)}{\prod_{k=1}^{n}(s+p_k)}$ *be a strictly proper causal transfer function. Assume there exists* $\psi \in (\pi/2, \pi]$ *such that* $arg(p_k) \in [\psi, 2\pi - \psi]$, $k = 1, 2, \ldots, n$. *Then, as* $\min_{k=1,2,\ldots,n} |p_k| \to \infty$, $\|F(s)\|_{\mathcal{L}_1} \to 0$.

Proof. See Appendix A.2. ■

Let us assume the filter $C(s)$ is chosen in the form of $1-s^n/(s^n+a_{n-1}s^{n-1}+\cdots+a_0)$, which guarantees that the DC gain of $C(s)$ is 1 and that the numerator of $1 - C(s)$

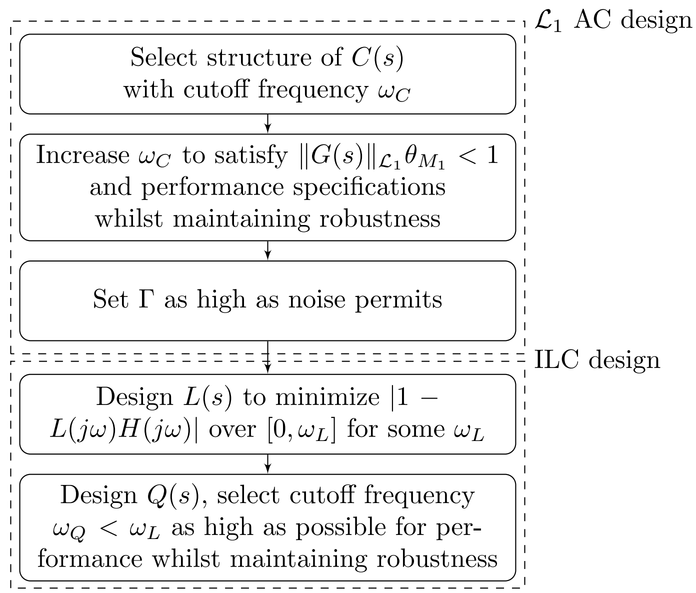


Figure 4.4: Design flowchart of the $\mathcal{L}_1$-ILC scheme

is constant regardless of the choice of poles. Now, we have

$$\|G(s)\|_{\mathcal{L}_1} \leq \left\| \frac{s^{n-1}}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \right\|_{\mathcal{L}_1} \|sH_x(s)\|_{\mathcal{L}_1}.$$

The above lemma then implies that $\|G(s)\|_{\mathcal{L}_1}$ can be rendered arbitrarily small to satisfy condition (4.2) by increasing the bandwidth of $C(s)$ since $\|sH_x(s)\|_{\mathcal{L}_1} \in \mathbb{R}$ from the stability assumption and the strict properness of $H_x(s)$. Observe that an obvious choice for $C(s)$ is $\omega_C/(s+\omega_C)$ and note that for a given $C(s)$ the $\mathcal{L}_1$ adaptive controller has guaranteed (bounded away from 0) robustness margins [110]. Hence, after $C(s)$ is designed, the adaptation rate should be set as high as possible, while taking into consideration that large values of Γ might amplify noise and hinder closed loop performance.

Once the adaptive control design is finalized, the learning function can be designed on the nominal system (i.e. $\theta = 0$) via the well known Nyquist tuning method [9]. A good rule of thumb to minimize the converged error is to set $|1 - L(j\omega)H(j\omega)|$ small within a large bandwidth since

$$e_\infty(s) = \frac{1 - Q(s)}{1 - Q(s)(1 - L(s)H(s))} e_{\text{fb}}(s),$$

for $\theta = 0$, where $e_{\text{fb}}(s)$ is the feedback error without any feedforward input. The Q filter can then be used to limit this bandwidth so that the learning controller is robust against unmodeled high frequency dynamics, noise, and the uncompensated parametric uncertainty $\Lambda(s)$ as per Theorem 4.4.

In light of these observations, the design procedure has been summarized in Figure 4.4. We remind the readers that while higher values of ω_C and ω_Q signify high closed loop and learning performance, this comes at the expense of reduced stability margins.

4.4 Output Feedback

The results of Section 4.3 show us that by a slight modification of the $\mathcal{L}_1$ AC formulation, we can preserve the guaranteed transient property of the feedback controller. By doing so, we make sure that the $\mathcal{L}_1$ controller uses information from the feedforward input and keeps the plant sensitivity close to the nominal case for performance improvement through learning. In this section we extend the results of Section 4.3 to the output feedback case with time varying unknown feedback gains and input disturbances. While the structure of the $\mathcal{L}_1$ controller is slightly different and less intuitive, we see that the results are similar from an ILC standpoint. We follow the same procedure of defining the feedforward augmented adaptive controller, and designing an iterative update law under the assumption of high adaptation gain. Unless explicitly stated, our assumptions and definitions from Section 4.3 will continue to hold.

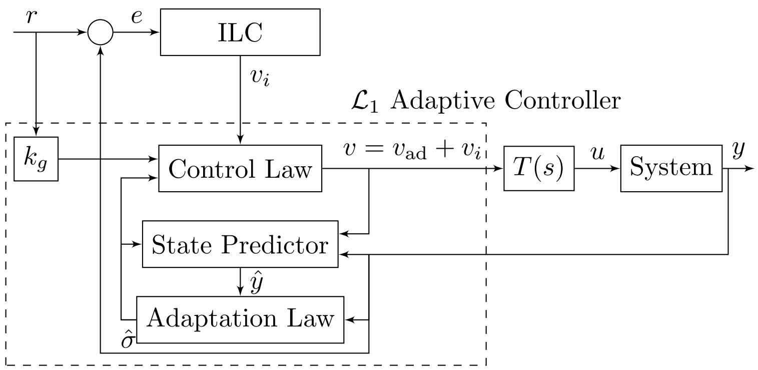


Figure 4.5: ILC with feedforward augmented $\mathcal{L}_1$ adaptive output feedback

4.4.1 $\mathcal{L}_1$ Adaptive Control

We present the $\mathcal{L}_1$ adaptive output feedback control architecture (Figure 4.5) for SISO linear systems with unknown time varying parameters and disturbances. Our main assumption is that the nominal system is minimum phase and of relative degree 1. The $\mathcal{L}_1$ controller for this class of systems considers an equivalent, *virtual system* with a virtual adaptive control input [117]. This virtual control signal is passed through a BIBO stable filter to synthesize the actual control input. Hence, we augment this virtual adaptive system with a virtual feedforward signal for learning purposes.

For completeness, we list some variables that are used in the analysis of the original controller [117] in Appendix A.3. We include some minor changes to account for the addition of an additional input in the adaptive controller.

4.4.1.1 Problem Formulation

Consider the class of systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b(u(t) + \theta^T(t)x(t) + \sigma(t)), \quad x(0) = x_{\text{in}}, \\ y(t) &= c^T x(t), \end{aligned} \tag{4.14}$$

where $x(t)$ is the *unmeasured* state vector; and $\sigma(t) \in \mathbb{R}$, $|\sigma(t)| \leq \Delta$ for some $\Delta \in \mathbb{R}^+$, is the time varying bounded disturbance.

Assumption 4.2. The transfer function $H(s)$ is minimum phase with relative degree 1.

Assumption 4.3. The signals $\theta(t)$ and $\sigma(t)$ are continuously differentiable with uniformly bounded derivatives; i.e. there exist $d_\theta, d_\sigma \in \mathbb{R}^+$ such that $\|\dot{\theta}(t)\|_2 \le d_\theta$ and $|\dot{\sigma}(t)| \le d_\sigma$.

The $\mathcal{L}_1$ AC objective is to track a given reference system in transient and steady state phases by using only output feedback.

4.4.1.2 System Transformation

In this section, we restate definitions and a lemma from [117] which will define our virtual system. Let

$$H_n(s) \triangleq b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n,$$

$$H_d(s) \triangleq s^n + a_1 s^{n-1} + \cdots + a_n,$$

where $a_k, b_k \in \mathbb{R}$ for $k = 1, 2, \ldots, n$ so that we have $H(s) = H_n(s)/H_d(s)$. Further let $A_T \in \mathbb{R}^{n\times n}$ such that the following equality holds:

$$H_x(s) = \frac{A_T \begin{bmatrix} 1 & s & \ldots & s^{n-1} \end{bmatrix}^T}{H_d(s)}.$$

Note that $H(s)$ is stable, minimum phase, and with relative degree 1 by assumption. Hence $H_n(s)$ and $H_d(s)$ are stable polynomials of order n and $n-1$, respectively, which implies $b_1 \neq 0$. Define

$$A_m \triangleq \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix},$$

$$b_m \triangleq \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T.$$

Since A_m is Hurwitz, for any $Z_m = Z_m^T > 0$ there exists $P_m = P_m^T > 0$ that solves

$$A_m^T P_m + P_m A_m = -Z_m.$$

Let $c_m \triangleq P_m b_m$. By the Kalman-Yakubovich-Popov lemma [44], the transfer function $H_m(s) \triangleq c_m^T (sI - A_m)^{-1} b_m = H_p(s)/H_d(s)$ is strictly positive real. For a given signal $v(s)$, let

$$u(s) = T(s)v(s), \tag{4.15}$$

where $T(s) \triangleq H_p(s)/H_n(s)$ with zero state space initialization. Further let $w_x(s)$ be the output of the following system $\mathcal{W}$:

$$\begin{aligned} w_x(s) &= wT^{-1}(s)w_1(s), \\ w_1(t) &= \theta^T(t)w_2(t), \\ w_2(s) &= T(s)A_T x(s). \end{aligned} \tag{4.16}$$

Lemma 4.5. *Given $v(t)$, $\theta(t)$, and $\sigma(t)$, there exists a signal $\sigma_m(t)$, $|\sigma_m(t)| \leq \Delta_m$ and $|\dot{\sigma}_m(t)| \leq d_{\sigma_m}$ for some $\Delta_m, d_{\sigma_m} \in \mathbb{R}^+$, such that the output $y(t)$ of (4.14) with input $u(t)$ synthesized according to (4.15) is equal to the output $y_m(t)$ of the following* *system:*

$$\dot{x}_m(t) = A_m x_m(t) + b_m(v(t) + w_{x_m}(t) + \sigma_m(t)), \quad x_m(0) = \hat{x}_{\text{in}},$$
$$y_m(t) = c_m^T x_m(t), \tag{4.17}$$

where $w_{x_m}(t)$ is the output of (4.16) with the input $x(t)$ replaced by $x_m(t)$ and $\hat{x}_{in}$ is any point such that we have $c_m^T \hat{x}_{\text{in}} = c_m^T x_{\text{in}}$ [117].

Since the above lemma states equivalence of the outputs for arbitrary $v(t)$, we can proceed with (4.17) as the actual system with proper modification of $v(t)$.

4.4.1.3 Closed Loop Reference System

With proper modification of $v_{ref}(t)$, the augmented closed loop reference system can be defined as

$$\begin{aligned}
&\dot{x}_{\text{ref}}(t) = A_m x_{\text{ref}}(t) + b_m(v_{\text{ref}}(t) + w_{x_{\text{ref}}}(t) + \sigma_m(t)), \quad x_{\text{ref}}(0) = \hat{x}_{\text{in}} \\
&v_{\text{ref}}(s) = C(s)\bar{r}_{\text{ref}}(s) + v_i(s), \\
&\bar{r}_{\text{ref}}(t) \triangleq k_g r(t) - w_{x_{\text{ref}}}(t) - \sigma_m(t), \quad k_g \triangleq 1/H_m(0), \\
&y_{\text{ref}}(t) = c_m^T x_{\text{ref}}(t),
\end{aligned} \tag{4.18}$$

where $w_{x_{\text{ref}}}(t)$ is the output of $\mathcal{W}$ with the input $x(t)$ replaced by $x_{\text{ref}}(t)$; $v_i(s)$ is an arbitrary bounded signal; and $C(s)$ is subject to the $\mathcal{L}_1$ norm condition

$$\|G_m(s)\|_{\mathcal{L}_1} M < 1, \tag{4.19}$$

where

$$G_m(s) \triangleq H_{xm}(s)(1 - C(s)),$$
$$H_{xm}(s) \triangleq (sI - A_m)^{-1} b_m,$$
$$M \triangleq \|T^{-1}(s)\|_{\mathcal{L}_1} \theta_{M_1} \|T(s) A_T\|_{\mathcal{L}_1}.$$

Lemma 4.6. *If (4.19) is satisfied, the reference system (4.18) is BIBS stable.*

Proof. See [117]. The proof follows in the same manner from the boundedness of $v_i(t)$. ■

Corollary 4.1. *For $\theta(t) = \theta$, the reference system (4.18) is BIBS stable if*

$$\|G_m(s)\|_{\mathcal{L}_1} \theta_{M_1} \|A_T\|_{\mathcal{L}_1} < 1. \tag{4.20}$$

Proof. The proof is omitted and is similar to the proof of Lemma 4.3. ■

4.4.1.4 $\mathcal{L}_1$ Adaptive Controller

The $\mathcal{L}_1$ adaptive controller for the virtual system is similar to that of the state feedback case, with the exception that we have a single adaptive law that estimates the combined effects of $w_{x_m}(t)$ and $\sigma_m(t)$.

State Predictor The controller has the following state predictor

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) + b_m(v(t) + \hat{\sigma}(t)), \quad \hat{x}(0) = \hat{x}_{\text{in}} \\ \hat{y}(t) &= c_m^T \hat{x}(t), \end{aligned} \tag{4.21}$$

where $\hat{y}(t)$ is the output prediction signal and $\hat{\sigma}(t)$ is the output of the adaptation law below.

Adaptation Law The adaptation law is given as

$$\dot{\hat{\sigma}}(t) = \Gamma_c \operatorname{Proj}(\hat{\sigma}(t), -\tilde{y}(t)), \quad \hat{\sigma}(0) = 0, \tag{4.22}$$

where $\tilde{y}(t) \triangleq \hat{y}(t) - y_m(t) = \hat{y}(t) - y(t)$ is the output prediction error; the projection is defined with the bound $\bar{\Delta}$ given in Section A.3; and Γ_c is the adaptation rate subject to

$$\Gamma_c > \max\left\{\frac{\alpha\beta_3}{(\alpha-1)^2\beta_4\lambda_{\min}(P_m)}, \frac{\alpha\beta_4}{\lambda_{\min}(P_m)\bar{\gamma}^2}\right\},$$

with $\alpha > 1$ arbitrary and $\beta_3, \beta_4, \bar{\gamma}$ defined in Section A.3.

Control Law The control law is given by

$$\begin{aligned} v(t) &= v_{\mathrm{ad}}(t) + v_i(t), \\ v_{\mathrm{ad}}(t) &\triangleq C(s)(k_g r(s) - \hat{\sigma}(s)), \end{aligned} \tag{4.23}$$

where $v_{\mathrm{ad}}(t)$ and $v_i(t)$ are the feedback and feedforward signals, respectively.

4.4.1.5 Transient Performance

The guaranteed transient property of the controller is given by the following theorem.

Theorem 4.5. *For system (4.17) with the controller by (4.21), (4.22) and (4.23), subject to the $\mathcal{L}_1$ norm condition (4.19); and its corresponding reference system (4.18), we have*

$$\begin{aligned} \|y_{\mathrm{ref}} - y_m\|_{\mathcal{L}_\infty} = \|y_{ref} - y\|_{\mathcal{L}_\infty} &\leq \frac{\gamma_1}{\sqrt{\Gamma_c}}, \\ \|v_{\mathrm{ref}} - v\|_{\mathcal{L}_\infty} &\leq \frac{\gamma_2}{\sqrt{\Gamma_c}}. \end{aligned} \tag{4.24}$$

Proof. See [117]. The proof follows the same structure with the redefinitions in Appendix A.3. ■

4.4.2 Iterative Learning Control

Having proved that the transient property holds with our additional feedforward signal, we are ready to design a learning law on the adaptive system for performance improvement. The recipe is the same as before and we will be using the nominal system to check robust monotonic convergence by bounding the system uncertainty.

4.4.2.1 Update Law

We use the Q filter and learning function approach as per Section 4.3 for simplicity and consistency with the state feedback case:

$$v_{i+1}(s) = Q(s)(v_i(s) + L(s)e_i(s)). \tag{4.25}$$

Note that since we consider (4.18) for design and analysis, we define the update law on the virtual control $v(s)$ as opposed to the actual control $u(s)$ (see Figure 4.5).

4.4.2.2 Monotonic Convergence and Robustness

We will design the learning controller under the same assumption as in the state feedback case; $x_m(t) = x_{ref}(t)$. We first analyze the case of constant feedback gain, i.e. $\theta(t) = \theta$. The closed loop reference system can then be described as

$$x_{m_i}(s) = H_{xm}(s)v_i(s) + H_{xm}(s)C(s)k_g r(s) + \\ G_m(s)\theta^T A_T x_{m_i}(s) + G_m(s)\sigma_m(s) + x_{\text{nr}}(s), \tag{4.26}$$

where $x_{\text{nr}}(s) \triangleq (sI - A_m)^{-1}\hat{x}_{\text{in}}$, which leads to

$$y_{m_i}(s) = \bar{H}_m(s)v_i(s) + \bar{H}_m(s)C(s)k_g r(s) + \\ \bar{H}_m(s)(1 - C(s))\sigma_m(s) + c_m^T(I - G_m(s)\theta^T A_T)^{-1}x_{\text{nr}}(s),$$

where $\bar{H}_m(s) \triangleq c_m^T(I - G_m(s)\theta^T A_T)^{-1}H_{xm}(s)$. The following extends the result of Theorem 4.4 to the output feedback case. Note that the condition is identical in structure to condition (4.9).

Theorem 4.6. *The ILC system with the update law (4.25) defined over $\bar{H}_m(s)$ subject* *to (4.19) or (4.20), is monotonically convergent with rate* $\mu_m \in [0,1)$ *for all* $\theta \in \Theta$ *if*

$$\kappa_m \leq \frac{\mu_m - |Q(j\omega)||1 - L(j\omega)H_m(j\omega)|}{|Q(j\omega)||L(j\omega)||H_m(j\omega)|}, \tag{4.27}$$

for all $\omega \in \mathbb{R}$*, where*

$$\kappa_m \triangleq \frac{\theta_{M_2}\|A_T G_m(s)\|_\infty}{1 - \theta_{M_2}\|A_T G_m(s)\|_\infty}.$$

Proof. The proof follows the same steps as that of Theorem 4.4 and is omitted. ■

We note that in both the state and output feedback cases, Theorems 4.4 and 4.6 show that the contraction mapping condition can be guaranteed for all uncertainties by well defined relationships that result from the $\mathcal{L}_1$ norm condition and bounds on the induced norms of the uncertainties. Therefore, we can extend the convergence conditions to time varying feedback in a similar fashion. To that end, we will rewrite the plant dynamics in operator form. Observe that in (4.26), the mapping $\theta^T A_T$ is in essence the system $\mathcal{W}$ that maps x_i to w_{x_i} for the special case of constant θ. Therefore, the plant dynamics can be rewritten in more general form as

$$x_{m_i} = \mathcal{H}_{xm} v_i + \mathcal{H}_{xm}\mathcal{C} k_g r + \mathcal{G}_m \mathcal{W} x_{m_i} + \mathcal{G}_m \sigma_m + x_{\text{nr}}, \tag{4.28}$$

where $\mathcal{H}_{xm}$, $\mathcal{C}$ and $\mathcal{G}_m$ are $H_{xm}(s)$, $C(s)$ and $G_m(s)$ in operator notation, respectively. Note that the dynamics are the same with the exception of $\mathcal{W}$ being a linear time varying map, which prevents us from further simplification in the Laplace domain. Regardless, we see after some manipulations that the $\mathcal{L}_2$ gain of the uncertainty and therefore the robust monotonic convergence condition is very similar.

Theorem 4.7. *The ILC system with the update law (4.25) defined over (4.28) subject to (4.19), is monotonically convergent with rate* $\mu_{m_{\text{tv}}} \in [0,1)$ *for all* $\theta(t) \in \Theta$ *if*

$$\kappa_{m_{\text{tv}}} \leq \frac{\mu_{m_{\text{tv}}} - \|Q(s)(1 - L(s)H_m(s))\|_\infty}{\|Q(s)L(s)H_m(s)\|_\infty}, \tag{4.29}$$

where

$$\kappa_{m_{\text{tv}}} \triangleq \frac{M_2\|T(s)A_T H_{xm}(s)\|_\infty \|1 - C(s)\|_\infty}{1 - M_2\|T(s)A_T G_m(s)\|_\infty},$$

with $M_2 \triangleq \|T^{-1}(s)\|_\infty \theta_{M_2}$.

Proof. See Appendix A.2. ■

Due to the time varying nature of the feedback uncertainty, Theorem 4.7 is naturally more conservative than Theorem 4.6. Algebraically, this is attested to the fact that we cannot simplify $\mathcal{W}$ and commute SISO operators as in matrix notation. Physically, we can interpret this as the effect of time varying parameters being much less predictable than that of constant parameters. Nevertheless, due to the condition being conservative, we might see in practice that the actual performance of ILC is much better than expected. We would also like to add that the design trade-offs of the output feedback $\mathcal{L}_1$-ILC scheme can be evaluated straightforwardly much as in Section 4.3.3. We omit these for the sake of brevity.

4.5 Simulations

To illustrate the benefits of our proposed method, we will consider an $\mathcal{L}_1$ AC based ILC design on a model of the flexure bearing based nanopositioner shown in Figure 4.6 [1]. In [1], the authors consider the following output compensator

$$D(s) = \frac{1.57 \times 10^4 (s + 141.5)(s^2 + 159.5s + 5.01 \times 10^4)}{s(s + 4000)(s^2 + 6700s + 1.92 \times 10^7)}, \tag{4.30}$$

designed on the open loop transfer function from the actuator input, identified as

$$P(s) = \frac{1.28 \times 10^{10}(s^2 + 5.63s + 3.34 \times 10^5)}{(s + 333.1)(s^2 + 150.50s + 3.31 \times 10^4)(s^2 + 12.43s + 3.87 \times 10^5)},$$

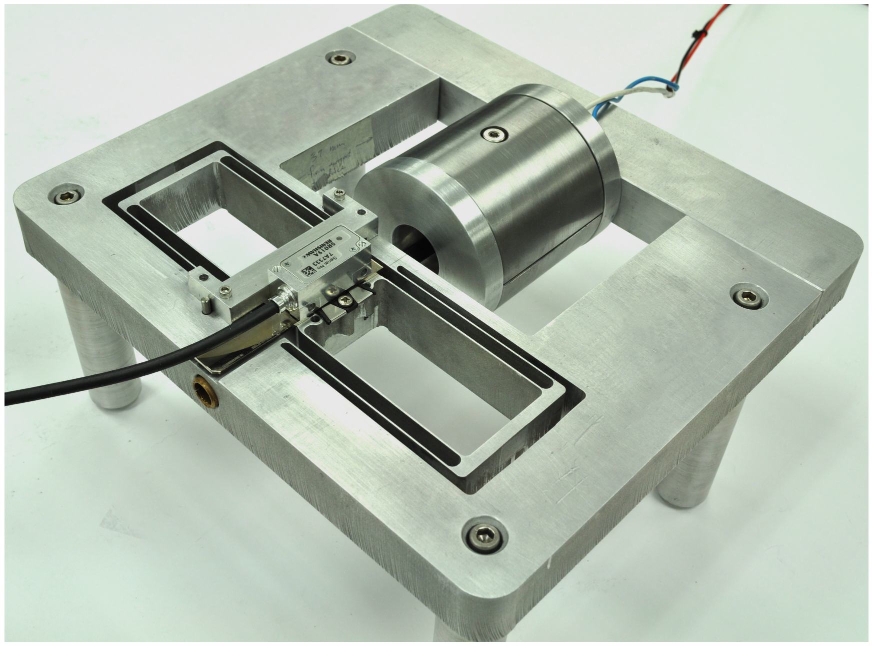


Figure 4.6: Single axis flexure bearing based nanopositioner with moving magnet actuator [1]

which results in the closed loop complementary sensitivity function

$$T_P(s) \triangleq D(s)P(s)/(1 + D(s)P(s)),$$

by unity gain feedback (Figure 4.7). Since $P(s)$ was obtained through system identification, we do a balanced realization to come up with the system matrices $\bar{A}$, b, and c such that $P(s) = c^T(sI - \bar{A})^{-1}b$. To simulate an uncertainty in the pole locations, we assume a time varying $\theta(t)$, subject to $\theta_{M_\infty} = 1$, with a bounded derivative.

We will be considering two feedback based learning schemes to compare on the plant $(\bar{A}, b, c^T, 0)$ with the uncertain feedback gain $\theta(t)$; LTI output feedback based ILC (LTI-ILC) and $\mathcal{L}_1$-ILC. The control objective is to minimize the tracking error for dynamic references within 10 rad/s, regardless of the level of uncertainty imposed by $\theta(t)$. We would like to see similar learning performances and converged tracking errors for every $\theta(t)$. Furthermore, we expect that performance degradations due to abrupt changes in $\theta(t)$ to be low, and can be compensated within a few iterations.

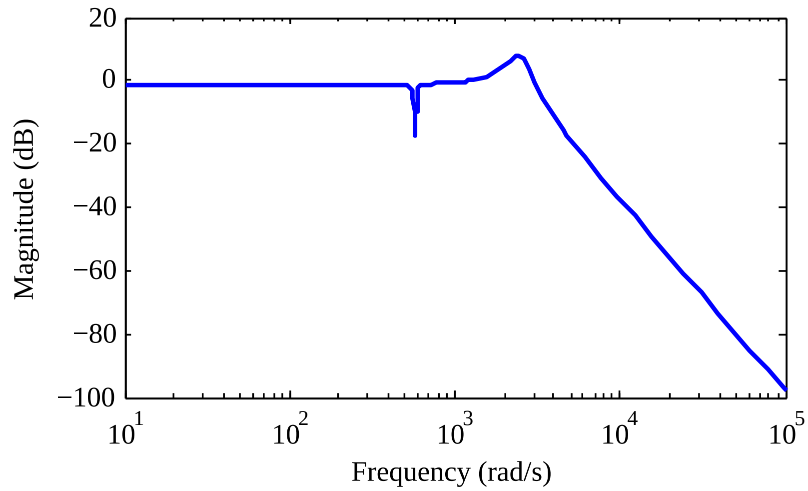


Figure 4.7: Bode magnitude plot of the closed loop complementary sensitivity function $T_p(s)$ of the flexure bearing based nanopositioner

4.5.1 LTI-ILC Design

We consider the output compensator of [1] and take the control law as

$$u_{\mathrm{LTI}_i}(s) = D(s)((r(s) + u_i(s)) - y_i(s)),$$

where $u_{\mathrm{LTI}_i}(s)$ is the feedback control input, $D(s)$ is defined in (4.30), and $u_i(s)$ is the feedforward learning signal. Note that this results in the dynamics given by $y_i(s) = T_p(s)u_i(s) + T_p(s)r(s)$, which is similar to the iteration domain dynamics in Section 4.3.2.2 under zero initial conditions and $\theta(t) = 0$.

The signal $u_i(s)$ is given by the update law (4.8), with

$$Q(s) = \frac{250}{s + 250}, \quad L(s) = \frac{5000}{s + 5000}.$$

The filter $L(s)$ is designed to approximate $T_p^{-1}(s)$ and to keep $|1 - L(j\omega)T_p(j\omega)|$ small over a large bandwidth, while $Q(s)$ is chosen to maintain stability whilst having a sufficiently high bandwidth. More specifically, the cutoff frequency of $L(s)$ was chosen to be higher than that of $T_p(s)$ for ample learning, while $Q(s)$ was chosen to be more than a decade faster than the desired tracking bandwidth of 10 rad/s. The signal $u_i(s) + L(s)e_i(s)$ is passed through $Q(s)$ twice via time reversal to emulate zero phase filtering in continuous time and eliminate the phase lag in the ILC signal. Note that the gain of this process is $|Q(j\omega)|^2$ due to double filtering.

4.5.2 State Feedback $\mathcal{L}_1$ AC based ILC Design

For the proposed $\mathcal{L}_1$-ILC scheme, we employ a static feedback k_{fb} such that the closed loop response given by $(A, b, c, 0)$, where $A = \bar{A} - bk_{\text{fb}}^T$, is similar to $T_p(s)$ under zero uncertainty, i.e. $\theta(t) = 0$. To ensure a fair comparison and have similar dynamics with the LTI case, we select the desired pole locations as the poles of the reduced order (5th) approximation to $T_p(s)$ to yield near identical step responses for $k_g H(s) = k_g c^T (sI - A)^{-1} b$ and $T_p(s)$.

In the $\mathcal{L}_1$-ILC design, we consider a 3rd order filter to better attenuate the effects of $\theta(t)$ and take $C(s) = 1 - s^3/(s+\omega_C)^3$, which satisfies (4.2) for $\omega_C \geq 2600$. Note that since the relative degree of the filter $C(s)$ is less than that of the plant $P(s)$, there exists no implementable LTI feedback controller equivalent to the resulting $\mathcal{L}_1$ AC reference control law (see Appendix A.4).

To avoid an excessive bandwidth, we choose $\omega_C = 3000$, and also note that condition (4.2) can be satisfied by a lower bandwidth through careful selection of the desired pole locations, since the lightly damped poles of $T_p(s)$ (and consequently $H(s)$ by virtue of the selected poles) manifest as high gain feedback through $\theta(t)$ in terms of the $\mathcal{L}_1$ norm condition. We consider a noisy measurement scenario wherein each state is corrupted by Gaussian white noise with variance 3.16×10^{-8}, which results in an output noise variance of 1.96×10^{-5}. To limit noise amplification, we take the adaptive gain Γ to be 1×10^6. For the state predictor, we select $K_{\text{sp}} = 0$ for simplicity.

For the ILC update law, we choose

$$Q(s) = \frac{250}{s+250}, \quad L(s) = k_g \frac{5000}{s+5000},$$

which result in a similar learning performance to that of the output feedback based design. As with the LTI-ILC scheme, we filter $u_i(s) + L(s)e_i(s)$ through $Q(s)$ twice to eliminate phase lag.

4.5.3 Simulation Setup

To compare the $\mathcal{L}_1$ AC and LTI based learning schemes, we will be looking at unknown parameters $\theta(t)$ with $\theta_1(t)$ as the only nonzero element. The reason for this is twofold: First, the Hankel singular values of the states x_1, x_2, x_3, x_4, and x_5 of the balanced realization of $P(s)$ are 910, 454, 172, 170, and 42; respectively. In other words, x_1 has a much higher contribution to the output when compared to other states. Second, by doing so we are able to consider large time varying uncertainties without having an excessively conservative robust monotonic convergence condition, stated as

$$\kappa_{\text{tv}} \leq \frac{\mu_{\text{tv}} - \|Q(-s)Q(s)(1 - L(s)H(s))\|_\infty}{\|Q(-s)Q(s)L(s)H(s)\|_\infty},$$

for $\mu_{\text{tv}} \in [0, 1)$, with

$$\kappa_{tv} \triangleq \frac{\theta_{M_\infty} \|1 - C(s)\|_\infty \|H_{x1}(s)\|_\infty}{1 - \theta_{M_\infty} \|G_1(s)\|_\infty},$$

where $H_{x1}(s)$ and $G_1(s)$ are the transfer functions to the first outputs of $H_x(s)$ and $G(s)$, respectively. The readers can verify that the condition guarantees monotonic convergence in the same vein as Theorems 4.4 and 4.7. More specifically, the terms $\|H_{x1}(s)\|_\infty$, $\|G_1(s)\|_\infty$, and θ_{M_∞} is used in the definition of κ_{tv} instead of $\|H_x(s)\|_\infty$, $\|G(s)\|_\infty$, and θ_{M_2} since $\theta(t)$ is zero for all elements but $\theta_1(t)$, and the inequalities $\theta_{M_\infty} < \theta_{M_2}$, $\|H_{x1}(s)\|_\infty \leq \|H_x(s)\|_\infty$, $\|G_1(s)\|_\infty \leq \|G_x(s)\|_\infty$ hold. Also note that this further implies $\theta_{M_\infty}\|G_1(s)\|_\infty \le \theta_{M_2}\|G(s)\|_\infty < 1$. On the other hand, the product $Q(-s)Q(s)$ is due to double filtering, where $Q(-s)$ is the stable anticausal counterpart of $Q(s)$.

For the simulation scenarios, we consider the responses of the two schemes to a sinusoidal reference, $r(t) = \sin(10t)$, over periods of 8 seconds, wherein each period defines a trial. At the beginning of each trial, we reset the clock to 0, and reinitiate the process with the updated feedforward signals. To better make our point, we consider *noiseless* measurements for the LTI feedback system.

4.5.4 Simulation Results

First, we look at the feedback response (without learning) of the two systems to a fast parameter, selected as $\theta_1(t) = \sin(50t)$. We see in Figure 4.8 that $\mathcal{L}_1$ AC clearly outperforms LTI control with an error norm (in the $\mathcal{L}_2$ sense) of 0.0126 against 0.138. We also observe that the $\mathcal{L}_1$ AC input is smooth and devoid of high frequency content from the estimation loop. Then, we study the learning performance of the two schemes and observe in Figure 4.9 that the $\mathcal{L}_1$-ILC scheme performs almost an order of magnitude better than the LTI feedback based system for all iterations. We note that, the converged error of the LTI scheme is much larger than that of the $\mathcal{L}_1$-ILC algorithm with significant effects due to the 50 rad/s feedback uncertainty. We look more closely at the converged error of the $\mathcal{L}_1$-ILC architecture in Figure 4.10 and see that the majority of the remaining error comprises Gaussian noise from the measurements.

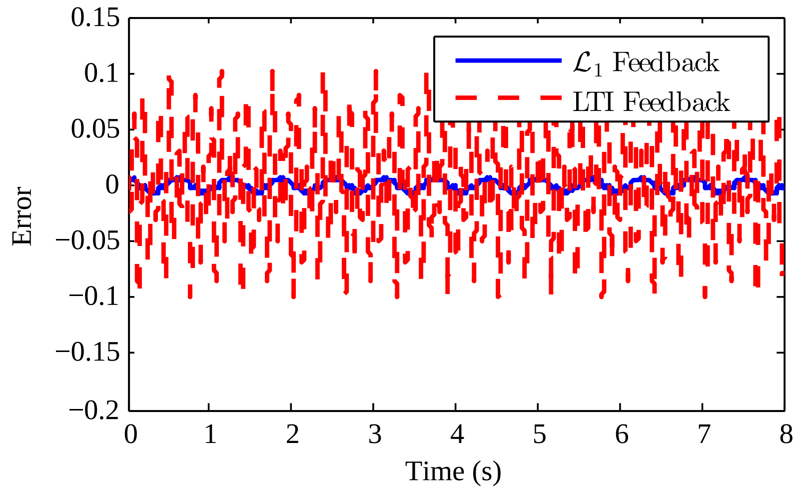
Next, we consider two scenarios wherein $\theta_1(t)$ has an abrupt change of sign at the 6th iteration. In the first scenario, we keep the parameter at the same frequency but decrease the amplitude to 0.1, thus starting the trials with $\theta_1(t) = 0.1\sin(50t)$ and switching to $\theta_1(t) = -0.1\sin(50t)$ at the 6th iteration. We see in Figure 4.11 that both controllers experience a large transient growth but converge back to equilibrium within a few trials. We also observe that the LTI feedback scheme performs better than before (compare to Figure 4.9) due to decreasing uncertainty, yet the learning performance is still poor when compared to $\mathcal{L}_1$-ILC with larger transients that exceed the original feedback control performance. In the second scenario, we assume a time invariant parameter with a very small amplitude and take $\theta_1(t) = 0.01$, which is changed to $\theta_1(t) = -0.01$ at the 6th iteration. Figure 4.12 shows us that the controllers show near identical learning dynamics due to the uncertainty being close to 0. We also see that the converged LTI error is slightly smaller than the $\mathcal{L}_1$ AC error due to the limiting noise factor. However, we observe that the error growth experienced by the LTI feedback based ILC is noticeable, whereas the $\mathcal{L}_1$-ILC system shows negligible change.

Finally, we redirect our attention to Figures 4.9, 4.11, and 4.12: We note that the $\mathcal{L}_1$-ILC scheme displays similar performance (in terms of initial and converged errors) regardles of the uncertainty, whereas the LTI feedback based ILC shows approximately an order of magnitude variance in terms of both the initial and converged errors. This clearly indicates the improvement in performance predictability for the $\mathcal{L}_1$-ILC system over the LTI feedback based ILC system.

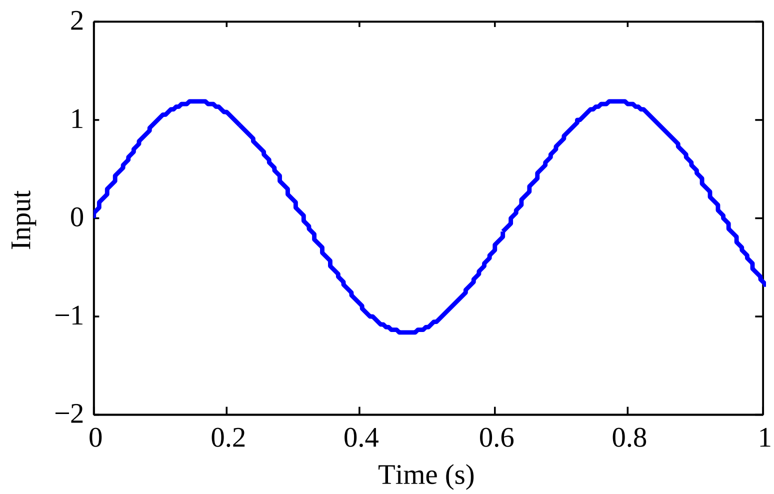
4.6 Conclusion

In this chapter we presented a combined $\mathcal{L}_1$-ILC scheme for robust precision motion control. $\mathcal{L}_1$ AC was utilized to reduce the effects of parametric variation and increase precision whilst preserving robustness against unmodeled dynamics. This reduction in parametric uncertainty enabled the use of aggressive ILC design to increase system bandwidth and improve tracking performance.

The combined controller is robust against parametric uncertainties *and* unmodeled dynamics, with *high tracking performance over a large bandwidth.* Simulation results on a precision nanopositioner demonstrate that the well posed feedback controller helps us in extracting high performance from ILC and achieve *near perfect tracking* even with information, bandwidth and hardware constraints, which is especially important due to the complex requirements for high precision tracking even in the presence of parametric uncertainty. It is important to note that the specific $\mathcal{L}_1$ AC architecture considered in the simulation scenario cannot be implemented as an equivalent LTI controller, since the relative degree of the filter is lower than that of the plant. Whether $\mathcal{L}_1$ AC has architectural advantages in the presence of unmodeled dynamics and signals is an issue that needs further attention. Similarly, whether lower relative degree filters have any benefits is an open question, as noted in Appendix A.4.

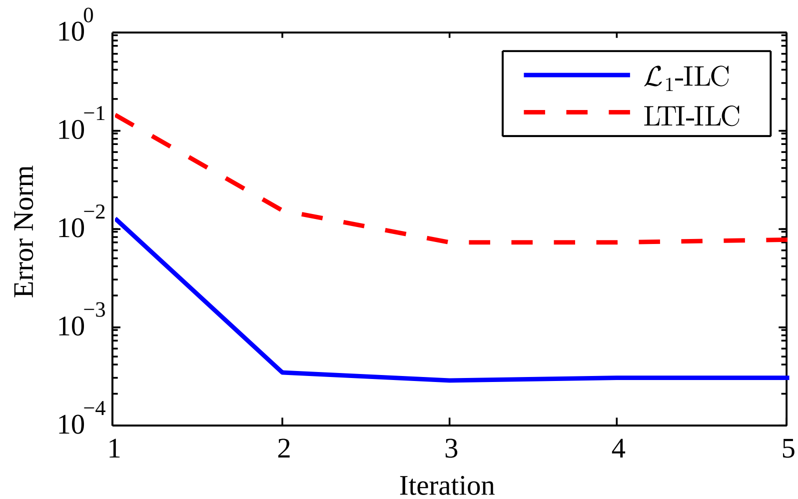


(a) Feedback tracking errors

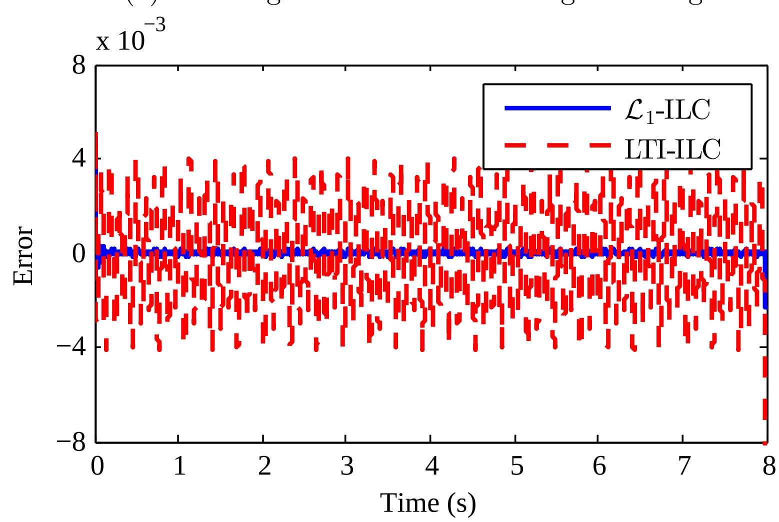


(b) Adaptive control input $u_{\text{ad}}(t)$ of the $\mathcal{L}_1$ adaptive controller

Figure 4.8: Closed loop responses of the $\mathcal{L}_1$ adaptive controller and LTI output controller to $\theta_1(t) = \sin(50t)$ without learning



(a) Tracking error reduction through learning



(b) Converged tracking errors

Figure 4.9: Learning performances of the $\mathcal{L}_1$-ILC and LTI-ILC schemes in response to the uncertain feedback $\theta_1(t) = \sin(50t)$

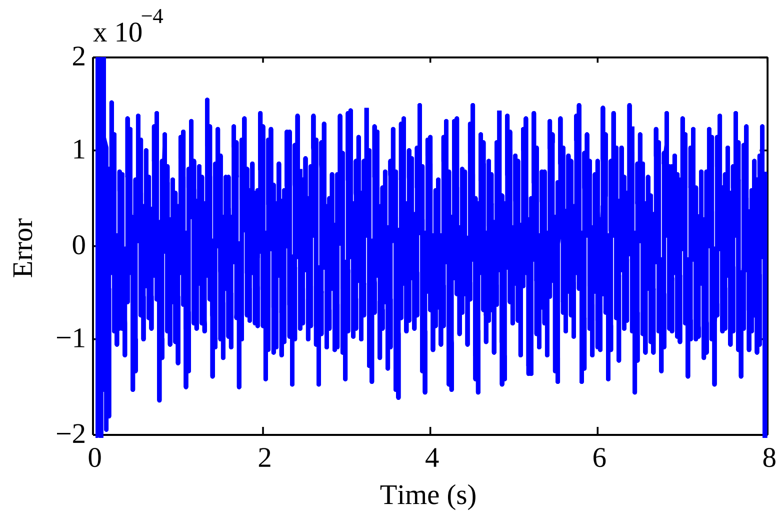


Figure 4.10: Converged tracking error of the $\mathcal{L}_1$-ILC scheme for $\theta_1(t) = \sin(50t)$

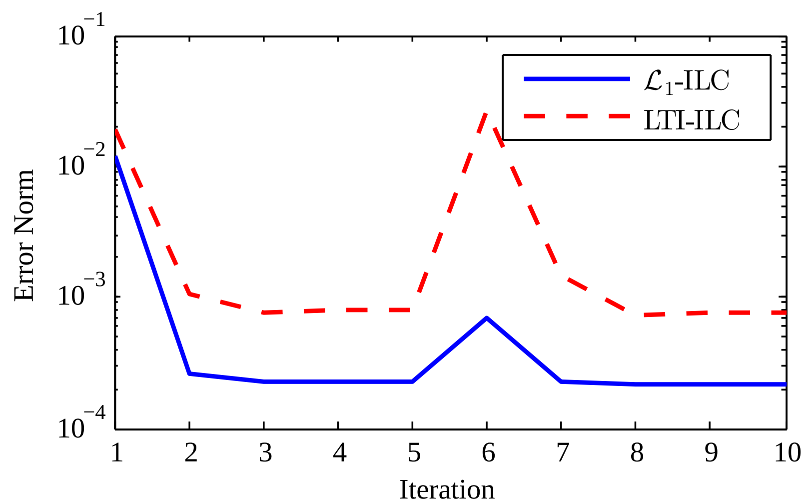


Figure 4.11: Learning transients of the $\mathcal{L}_1$-ILC and LTI-ILC schemes due to an abrupt change in the sign of $\theta_1(t) = 0.1\sin(50t)$ at the 6th iteration

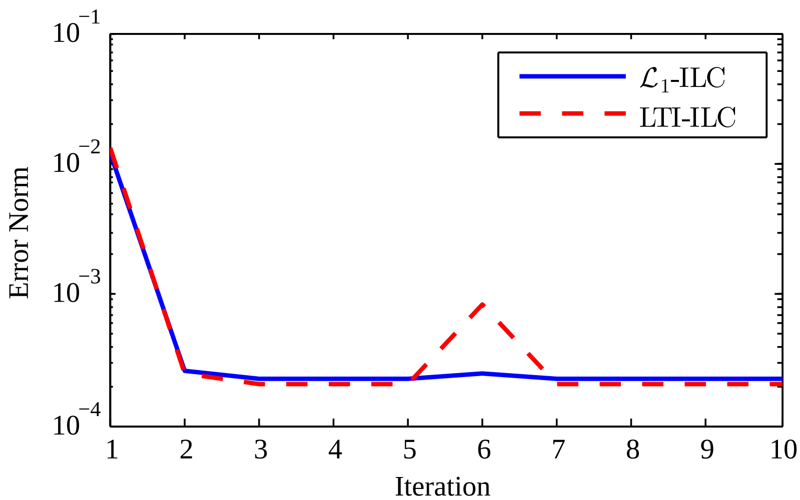


Figure 4.12: Learning transients of the $\mathcal{L}_1$-ILC and LTI-ILC schemes due to an abrupt change in the sign of $\theta_1(t) = 0.01$ at the 6th iteration

CHAPTER 5

Robust Stability of Iteration Varying Systems with Experimental Implementation

In the previous chapter, we considered the robust monotonic convergence problem of uncertain linear systems for high precision tracking performance, targeting the application area of precision motion control systems. Although the analysis assumed iteration invariance of the uncertainties, we were motivated by relaxing the plant invariance assumption a la adaptive feedback control. In this chapter, we consider the stability and convergence of linear iteration varying systems. In the introduction to this chapter, we recall some of our arguments from earlier about the invariance assumption and the feedback analogy of ILC.

5.1 Introduction

The fundamental assumption that enables the success of ILC algorithms is the iteration invariance of the: 1) plant dynamics, 2) exogenous disturbances, 3) initial conditions, and 4) reference. This assumption greatly simplifies the ILC problem and enables the control engineer to design an asymptotically stable recurrence relation in the iteration domain by employing a contraction mapping. Even though the

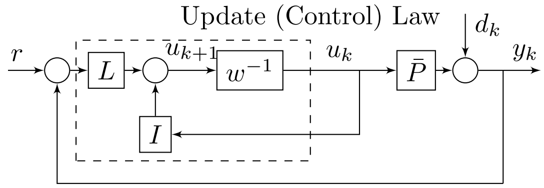


Figure 5.1: Feedback control in the iteration domain interpretation of ILC: The inherent integral action of the control law ensures that the output y_k converges to r for constant $d_k = d$ for the static LTI plant $\bar{P}$, provided the feedback loop is stable. Here, w^{-1} represents the trial delay operator, and d_k is a term that represents disturbances, noise, and the effect of initial conditions.

assumption is unrealistic, similar to feedback control of LTI systems, it yields good results in practice provided that the variation of the process (dynamics, exogenous disturbances, initial conditions etc.) from trial to trial is small.

5.1.1 The Feedback Analogy

The restrictive nature of the invariance assumption is perhaps best understood via an analogy to feedback control, since a common interpretation of ILC is that of a feedback controller in the iteration domain, as per the following discussion: Let $\bar{P} : U \to Y$ be a bounded linear operator, where U is the space of admissible inputs and Y is the space of outputs. Assuming that $\bar{P}$ is known and there are no exogenous signals apart from u_k affecting the output, the classical ILC problem can be stated as that of finding a controller C that maps the input history $u_0, u_1, \ldots, u_{k-1} \in U$ to the current input u_k, such that the output $y_k = \bar{P}u_k$ converges to a desired reference r in the image of $\bar{P}$ as $k \to \infty$. In most cases, C is designed to consider only the previous iteration, thus giving rise to the name first order ILC. The internal model principle then dictates that the controller (update law) C includes integral action to guarantee perfect tracking in the limit, so $C(u_k) = u_{k-1} + L(r - \bar{P}u_{k-1})$, as can be seen in Figure 4.2, which guarantees $y_k \to r$ even in the case where the output is corrupted by a constant vector $d \in Y$ such that $y_k = \bar{P}u_k + d$. Essentially, the ILC problem is that of designing a time invariant feedback controller for a constant static plant to track step references [7], under the assumption of constant noise and disturbance signals.

The objective of this chapter is to generalize the ILC problem by relaxing the invariance assumption, which restricts the feedback analogy to setpoint tracking, and fails to capture the generality associated with the feedback paradigm. In practice, initial conditions and disturbances are *always subject to variations*, while references and plants can commonly appear as outputs of HOIMs in the context of robotic manipulators doing different tasks, or freeway traffic models [118].

5.1.2 Relevant Work

Linear feedback control encompasses a wide array of problems and their accompanying solutions, such as stabilization, robustness, optimality, sensitivity reduction, fundamental limitations, and design trade-offs. Since the 1990s, there has been an increased effort in the ILC community to generalize the classical problem in these directions. These include the synthesis of 1) robust ILC algorithms [32,36,69–73], 2) norm optimal ILC algorithms with quadratic cost functions, 3) AILC methodologies [80,82–84], along with the study of performance guidelines and design trade-offs [32, 101, 102]. See also [9, 12, 33] and the references therein.

Implicit in the vast majority of these earlier works is the invariance assumption in some form. To date, there has been relatively limited material attempting to relax these assumptions. Among these, initial condition invariance was by far the most discussed topic earlier in the literature, since perfect resetting can be hard to achieve for certain systems [103]. The central result of [103] shows that initial condition resetting errors and bounded disturbances affect the tracking error continuously, provided they are uniformly bounded in the iteration domain. The effects of varying disturbance signals have been studied in stochastic settings [32, 69, 71, 119]. Varying references are also increasingly studied in ILC theory; AILC is one of the avenues in which this objective is pursued [33, 81], while some other works consider parametrizing the set of references by basis functions [34, 104, 105, 120] or library based interpolations [35]. Lastly, iteration varying plant models are actively studied in the case that they can be described by a HOIM [107], with generalizations to iteration varying references and signals considered in [121].

Despite all these efforts, the feedback interpretation of ILC still paints mostly an incomplete picture, and lacks the fundamental notions of *asymptotic and input-output stability*. In this sense, the introduction of the w transform (z transform in the iteration domain) in [50] has been crucial in adopting a more holistic view of ILC as an input-output system, induced by feedback control in the iteration domain. The transform enables the integration of iteration varying signals into the ILC problem and is a good step towards the establishment of input-output stability properties in ILC. However, it restricts the analysis to iteration invariant plants and update laws. On the other hand, while [75] presents a framework to investigate the stability of discrete time iteration varying systems, the analysis is restricted to iteration invariant signals.

Our aim in this chapter is 1) to construct a general framework to analyze stability properties of ILC systems in the presence of iteration varying signals (including references) and plant operators, where the operators are assumed to belong to a bounded subset and otherwise unknown, and 2) connect our analysis to the robust ILC literature by showing that robust updates lead to stable behavior in ILC. In addition, we will compare the performance of this uncertain iteration varying system to its nominal invariant counterpart, discuss how nominal performance can be recovered, and verify the theory with simulation examples and experimental implementation.

5.1.3 Organization of the Chapter

The remainder of the chapter is organized as follows: Section 5.2 introduces preliminaries and the ILC problem. Section 5.3 proves the basic boundedness result of the algorithm. In Section 5.4, asymptotic performance and design trade-offs are investigated. Section 5.5 describes the experimental setup, which also forms the basis for the simulation examples. Simulation examples are presented in Section 5.6, with the experimental results following in Section 5.7. Finally, concluding remarks are given in Section 5.8.

5.2 Problem Formulation

We recall the vector space formulation of the first order ILC problem presented in Chapter 2, also discussed in Section 5.1. We assume U and Y to be Banach spaces equipped with suitable norms. We base this assumption on the fact that Banach spaces are the natural settings of contraction mapping based ILC, which relies on the fixed point theorem. Furthermore, $\mathcal{L}_p$ and l_p spaces, the natural framework for 1D dynamic systems, are complete. The motivation for this assumption is to come up with a general framework that contains the variety of different settings in ILC, consistent with the vector space approach in [7].

In this framework, the operator $\bar{P}$ represents the input-output relationship of a linear system, which can be described by an ordinary differential equation, a partial differential equation, or a difference equation, over a finite or infinite domain. By taking U and Y as complete normed spaces, and $\bar{P} : U \to Y$ as a bounded linear operator, we will be able to have a complete analysis valid for a broad class of problems, in a simplified fashion.

Care must be taken in the definitions of the operators, as boundedness depends on the specific choice of spaces. Two examples are given below.

Example 5.1. Consider the scalar differential equation $\dot{y}(t) = cy(t) + u(t)$ with the initial condition $y(0) = 0$, where $c \in \mathbb{R}$, and $U = Y$ is the space of continuous functions over the interval $[a, b]$ with the sup norm. The differential equation is a bounded operator for

- $a = 0$, $b = \infty$, if and only if $c < 0$,
- $a = -\infty$, $b = 0$, if and only if $c > 0$, and
- $a, b \in \mathbb{R}$, for all c.

Example 5.2. Consider the convolution operator represented by the transfer function $1/(1 - s)$. This operator is unbounded (unstable) if the transfer function is the one sided Laplace transform (over the positive real line) of the kernel e^t, but is bounded (stable) if it is the bilateral Laplace transform of the kernel $e^t\mathbf{1}(-t)$, where $\mathbf{1}(.)$ is the Heaviside step function.

5.2.1 Notation and Preliminaries

We take $\mathbb{N}$ to represent the set of nonnegative integers and $\mathbb{N}^+$ the set of positive integers. For normed vector spaces X and V, $B(X, V)$ is the space of all bounded linear operators from X to V. We use $\|.\|$ to denote vector and induced operator norms in the relevant spaces. For a family of operators indexed by a subset of $\mathbb{N}$, the product notation indicates the composition of the operators in increasing order; e.g. $\prod_{i=j}^{k} H_i \triangleq H_k H_{k-1} \dots H_j$ for $j \leq k$ and $\prod_{i=j}^{k} H_i \triangleq I$ for $j > k$, where I is the identity. The uniform distribution over $[a, b]$ is denoted $\mathcal{U}(a, b)$.

For a rigorous study of the convergence and stability of the iterative problem, we define the spaces[20] $U^\omega \triangleq \prod_{k\in\mathbb{N}} U$ and $Y^\omega \triangleq \prod_{k\in\mathbb{N}} Y$. An element x in these spaces will be defined so x_k denotes the kth coordinate. We will use this notation to refer to any sequence of objects in the same space, e.g. $x \triangleq (x_0, x_1, \dots)$ where each x_k can be an element of U, Y, or an operator in these spaces. In addition, we introduce the following definitions where the spaces X and V are in $\{U, Y\}$.

[20]The notation $\mathbb{R}^\omega$ typically denotes the product $\prod_{k\in\mathbb{N}} \mathbb{R}$, hence U^ω, Y^ω.

Definition 5.1. Let x be an element of X^ω. The norm of x is given by $\|x\| \triangleq \sup_{k\in\mathbb{N}} \|x_k\|$.

Definition 5.2. A linear mapping $H : X^\omega \to V^\omega$ is *BIBO stable* if there exist a finite constant ϵ such that $\|(Hx)_\kappa\| \leq \epsilon\|(x)_\kappa\|$ for all $x \in X^\omega$ and $\kappa \in \mathbb{N}$, where $(x)_\kappa \triangleq (x_0, x_1, \dots, x_\kappa, 0, 0, \dots)$ is the truncation of x.

The spaces U^ω and Y^ω are not normed spaces since our definition of the norm entails the possibility of unbounded elements. However, this is merely a formality and will not affect our analysis as any truncated vector in these spaces has a finite norm. This is akin to the definition of input-output stability via the extended space $\mathcal{L}_{pe}$.

Definition 5.3. Let $x, v \in X^\omega$. We say x *converges to* v if $\lim_{k\to\infty} \|x_k - v_k\| = 0$. Otherwise, if $\limsup_{k\to\infty} \|x_k - v_k\|$ is bounded, we say x *converges to a bounded neighborhood of* v.

Definition 5.4. Let $H_k \in B(X, X)$. The system defined by the equality $x_{k+1} = H_k x_k$ for all $k \in \mathbb{N}$ is *asymptotically stable* if there exists a scalar ϵ such that $\|x\| \leq \epsilon\|x_0\|$, and x converges to 0 for all $x_0 \in X$.

The framework described above will enable us to adopt a *holistic signal space approach to ILC*, with the closed loop system (in the iteration domain) as the input-output operator, so stability and convergence can be studied for the case of iteration varying factors.

5.2.2 System Dynamics

Based on the above, we consider the following class of systems:

$$y_k = P_k u_k + d_k, \quad \forall k \in \mathbb{N}, \tag{5.1}$$

where $y_k \in Y$ is the output, $u_k \in U$ is the input, $d_k \in Y$ is the exogenous signal that includes disturbance, noise, and the effect of initial conditions, and P_k is the iteration varying linear input-output operator. Moreover, we assume that each P_k is in the vicinity of the known bounded linear operator $\bar{P}$ as stated in the following assumption.

Assumption 5.1. The input-output operators lie in a neighborhood of $\bar{P}$. In other words, there exists a finite real constant ρ such that

$$P_k \in \mathcal{P} \triangleq \left\{H \in B(U,Y) : \|H - \bar{P}\| < \rho\right\}, \quad \forall k \in \mathbb{N}.$$

Due to the assumption that the process variables P_k and d_k are varying along the iteration axis, it is a straightforward matter to assume that the reference is also subject to variations from trial to trial. Thus, our objective is to solve the following problem:

Problem 5.1. Find an ILC update law such that the error vector e defined by $e_k \triangleq r_k - y_k$ for all $k \in \mathbb{N}$, where the reference r_k is in the image of $\bar{P}$ for all $k \in \mathbb{N}$, converges to a small neighborhood of 0.

As with the plant operators, we make a boundedness assumption on r.

Assumption 5.2. The reference vectors lie in a neighborhood of a nominal reference $\bar{r}$ in the image of $\bar{P}$. In other words, there exists a finite real constant ζ such that

$$r_k \in \mathcal{R} \triangleq \left\{h \in \bar{P}(U) \subset Y : \|h - \bar{r}\| < \zeta\right\}, \quad \forall k \in \mathbb{N}.$$

5.3 Stability of Iteration Varying Systems via Robust Update Laws

This section will detail the stability analysis of our proposed solution to Problem 5.1. The solution will generalize the findings of [75] along the abstract contraction mapping approach of [7], and connect the iteration varying problem to the robust ILC literature. Consider the most general linear iteration invariant update law

$$u_{k+1} = Qu_k + Le_k, \quad \forall k \in \mathbb{N}, \tag{5.2}$$

where Q and L are bounded, and u_0 is arbitrary. Furthermore, we will require the update law to be subject to the robustness condition

$$\|Q - LH\| \leq \gamma < 1, \quad \forall H \in \mathcal{P}, \tag{5.3}$$

for some real constant γ, which guarantees monotonic convergence for all $H \in \mathcal{P}$ when the system is iteration invariant.

Condition 5.3 is a sufficient condition for asymptotic stability of the iteration varying input equation, as we shall see below. When the spaces U and Y are finite dimensional, i.e. Q, L, and all $H \in \mathcal{P}$ have matrix representations, (uniform/robust) asymptotic stability is equivalent to the *joint spectral radius* of the bounded set of operators $(Q - L\mathcal{P})$ being strictly less than 1, which has been shown to be an undecidable problem [122].

Substituting (5.1) into the update law (5.2) yields the recurrence relation

$$u_{k+1} = T_k u_k + L\eta_k, \quad \forall k \in \mathbb{N}, \tag{5.4}$$

where $T_k \triangleq Q - LP_k$ and $\eta_k \triangleq r_k - d_k$. The solution of the input vector in terms of u_0 and η_k can then be given as

$$u_{k+1} = \underbrace{\left(\prod_{i=0}^{k} T_i\right) u_0}_{\text{Natural response}} + \underbrace{\sum_{i=0}^{k}\left(\prod_{j=i+1}^{k} T_j\right) L\eta_i}_{\text{Forced response}}, \quad \forall k \in \mathbb{N}. \tag{5.5}$$

Equation (5.4) defines a time (iteration) varying discrete dynamical system on the space U, as such, its solution (5.5) is conceptually the same as that of a discrete time system on $\mathbb{R}^n$. When (5.3) holds, since $\|T_k\| \leq \gamma < 1$, it is easy to see that (5.4) is a well-defined, stable dynamical system.

Proposition 5.1. *The linear iterative system described by* (5.4) *with* $\eta = 0$*, subject to* (5.3)*, is asymptotically stable.*

Proof. Assume (5.3) holds and $\eta = 0$. Take an arbitrary $u_0 \in U$. Then from (5.5), $\|u_{k+1}\| \leq \gamma^{k+1}\|u_0\|$. Since $\gamma < 1$, it follows that u converges to 0 and $\|u\| \leq \|u_0\|$. Therefore, system (5.4) is asymptotically stable. ■

Proposition 5.2. *The linear iterative system described by* (5.4) *with input* η*, subject to* (5.3) *and the equality* $u_0 = 0$*, is BIBO stable.*

Proof. Assume (5.3) holds and $u_0 = 0$. Take any $\eta \in Y^\omega$. Then from (5.5) we have

$$\begin{aligned}
\|u_{\kappa+1}\| &\leq \sum_{i=0}^{\kappa} \gamma^{\kappa-i}\|L\|\|(\eta)_\kappa\| = \frac{1-\gamma^{\kappa+1}}{1-\gamma}\|L\|\|(\eta)_\kappa\| \\
&\leq \frac{\|L\|\|(\eta)_\kappa\|}{1-\gamma} \leq \frac{\|L\|\|(\eta)_{\kappa+1}\|}{1-\gamma}, \quad \forall \kappa \in \mathbb{N},
\end{aligned}$$

where we use the fact that the truncated norm is monotonically increasing by definition. Using the same property, we can show by the above inequality that

$$\|(u)_\kappa\| = \max_{i=1,\dots,\kappa} \|u_i\| \leq \frac{\|L\|\|(\eta)_\kappa\|}{1-\gamma}, \quad \forall \kappa \in \mathbb{N}. \tag{5.6}$$

Therefore, system (5.4) is BIBO stable. ■

We showed that the recursion relation (5.4) is asymptotically and BIBO stable when subject to (5.3). We finish this section with the following theorem, which shows that u and y are bounded if d is bounded.

Theorem 5.1. *The signals u and y of the linear iterative system* (5.1) *with the update law* (5.2) *is bounded if d is bounded.*

Proof. Consider the solution (5.5) of the input u, which is the superposition of the *natural response* describing the asymptotic response to the initial condition u_0, and the *forced response* describing the input-output behavior due to η. Since r is bounded by Assumption 5.2, η is bounded if d is bounded. From Propositions 5.1 and 5.2, it follows that u is bounded. Now observe that

$$\|y_k\| \leq \|P_k\|\|u_k\| + \|d_k\| \leq \|P_k\|\|u\| + \|d\|, \quad \forall k \in \mathbb{N},$$

by (5.1). Since $\mathcal{P}$ is uniformly bounded, it follows that y is bounded. ∎

The results of this section show that an ILC update law can be safely applied on iteration varying systems, provided the update law is designed to be robust against plant uncertainties. Based on the nature of the underlying spaces U, Y, and the operator set $\mathcal{P}$, this update law can be designed using existing robust ILC techniques.

5.4 Asymptotic Performance and Design Trade-offs

Having shown that the ILC system with our proposed solution is well-posed under the robustness assumption, we will direct our attention to the asymptotic performance of the system, when compared to a nominal iteration invariant systems. One motivation for analyzing these systems in general, as opposed to systems where $Q = I$, is that perfect tracking can be an infeasible objective for various reasons. For example, the set $\mathcal{P}$ might be too big, so (5.3) cannot be satisfied for $Q = I$. As such, we will introduce a nominal iterative system via the known operator $\bar{P}$ and reference $\bar{r}$ under the assumption that $d = 0$, which will facilitate our analysis.

5.4.1 Asymptotic Response of the System and the Corresponding Nominal Dynamics

As the choice of u_0 has no effect on the input (5.5) as the iteration index $k \to \infty$, we will drop the natural response from (5.5), and consider

$$\begin{aligned} u_{k+1} &\triangleq \sum_{i=0}^{k} \left(\prod_{j=i+1}^{k} T_j \right) L\eta_i, \\ e_k &\triangleq -P_k u_k + \eta_k, \end{aligned} \tag{5.7}$$

for all $k \in \mathbb{N}$, where $u_0 = 0$.

We define the nominal asymptotic system to be the case where the signal $d = 0$ and the plant $P_k = \bar{P}$ for all $k \in \mathbb{N}$. In other words, we describe the nominal system as

$$\bar{y}_k = \bar{P}\bar{u}_k, \quad \forall k \in \mathbb{N},$$

where $\bar{y}_k \in Y$ is the nominal output and $\bar{u}_k \in U$ is the nominal input. Thus, the error dynamics of the nominal system are given by the relation below, where $\bar{\eta} \triangleq \bar{r}$:

$$\bar{e}_k = -\bar{P}\bar{u}_k + \bar{\eta}, \quad \forall k \in \mathbb{N}.$$

We take the update law as $\bar{u}_{k+1} = Q\bar{u}_k + L\bar{e}_k$, with Q and L the same as before.

Consequently, since the choice of $\bar{u}_0$ has no effect in the limit, we consider

$$\begin{aligned}\bar{u}_{k+1} &\triangleq \sum_{i=0}^{k}\left(\prod_{j=i+1}^{k}\bar{T}\right)L\bar{\eta},\\ \bar{e}_k &\triangleq -\bar{P}\bar{u}_k+\bar{\eta},\end{aligned}\tag{5.8}$$

for all $k \in \mathbb{N}$, where $\bar{T} \triangleq Q - L\bar{P}$ and $\bar{u}_0 = 0$. This nominal system is well known to be stable and convergent, with the limits $\bar{u}_\infty \triangleq \lim_{k\to\infty} u_k$ and $\bar{e}_\infty \triangleq \lim_{k\to\infty} e_k$, when (5.3) holds.

5.4.2 Asymptotic Learning Performance

We will now analyze the performance of the algorithm (5.2) on the ILC system. Towards that end, based on the results of the previous section, we will compare the dynamics (5.7) and (5.8) written below in recursive form:

$$\bar{u}_{k+1} = \bar{T}\bar{u}_k + L\bar{\eta}, \quad \forall k \in \mathbb{N}, \tag{5.9}$$

$$u_{k+1} = T_k u_k + L\eta_k, \quad \forall k \in \mathbb{N}. \tag{5.10}$$

The equalities above will enable us to show that the iteration varying ILC system converges to a bounded neighborhood of the nominal invariant system. In showing this result, the main idea is to subtract the system (5.10) from the nominal dynamics (5.9) and come up with a stable recursion, driven by the bounded uncertainties due to P, r, d.

Theorem 5.2. *Assume that the linear iterative system described by* (5.1) *with the update law* (5.2) *is subject to* (5.3). *Then, if d is bounded, u and e converge to a* *neighborhood of $\bar{u}$ and $\bar{e}$, respectively. In other words,*

$$\limsup_{k\to\infty} \|\tilde{u}_k\| \leq \|L\| \frac{\rho\|\bar{u}_\infty\| + \zeta + \|d\|}{1-\gamma}, \tag{5.11}$$

and

$$\limsup_{k\to\infty} \|\tilde{e}_k\| \leq \left(\|L\| \frac{\|\bar{P}\| + \rho}{1-\gamma} + 1 \right) (\rho\|\bar{u}_\infty\| + \zeta + \|d\|). \tag{5.12}$$

Proof. Assume that (5.3) holds. Then, subtracting (5.10) from (5.9), it is easy to show

$$\|\tilde{u}_{k+1}\| \leq \gamma\|\tilde{u}_k\| + \|L\|(\|\tilde{P}_k\|\|\bar{x}_k\| + \|\tilde{r}_k\| + \|d_k\|), \tag{5.13}$$

for all $k \in \mathbb{N}$, where $\tilde{x}_k \triangleq \bar{x}_k - x_k$, $\tilde{P}_k \triangleq \bar{P} - P_k$, and $\tilde{r}_k \triangleq \bar{r} - r_k$. Now recall that $\bar{u}$ converges to a fixed point $\bar{u}_\infty$. Hence

$$\limsup_{k\to\infty} \|\tilde{u}_k\| \leq \gamma \limsup_{k\to\infty} \|\tilde{u}_k\| + \|L\|(\rho\|\bar{u}_\infty\| + \zeta + \|d\|),$$

for all $k \in \mathbb{N}$. It follows that

$$\limsup_{k\to\infty} \|\tilde{u}_k\| \leq \|L\| \frac{\rho\|\bar{u}_\infty\| + \zeta + \|d\|}{1-\gamma},$$

which verifies (5.11). Similarly, letting $\tilde{e}_k \triangleq \bar{e}_k - e_k$ for all $k \in \mathbb{N}$, it is easy to derive

$$\tilde{e}_k = P_k\tilde{u}_k - \tilde{P}_k\bar{u}_k + \tilde{r}_k + d_k, \quad \forall k \in \mathbb{N},$$

which leads to the inequality

$$\|\tilde{e}_k\| \leq (\|\bar{P}\| + \|\tilde{P}_k\|)\|\tilde{u}_k\| + \|\tilde{P}_k\|\|\bar{u}_k\| + \|\tilde{r}_k\| + \|d_k\|, \tag{5.14}$$

for all $k \in \mathbb{N}$. From above, by substituting (5.11) it follows that

$$\limsup_{k\to\infty} \|\tilde{e}_k\| \le \left(\|L\| \frac{\|\bar{P}\| + \rho}{1-\gamma} + 1 \right) (\rho \|\bar{x}_\infty\| + \zeta + \|d\|),$$

which verifies (5.12). This completes the proof. ■

In addition, if the input-output operator and the reference converge to the nominal case, and d converges to 0, it can be shown that the ILC system converges to the nominal invariant system, as shown in the following theorem. Here, convergence of P to $\bar{P}$ is to be interpreted as $\lim_{k\to\infty} \|P_k - \bar{P}\| = 0$ as in Definition 5.3.

Theorem 5.3. *Assume that the linear iterative system described by* (5.1) *with the update law* (5.2) *is subject to* (5.3)*. Then, if P converges to $\bar{P}$, r converges to $\bar{r}$, and d converges to 0, u and e converge to $\bar{u}$ and $\bar{e}$, respectively.*

Proof. Consider (5.13). Then we have

$$\limsup_{k\to\infty} \|\tilde{u}_k\| \le \gamma \limsup_{k\to\infty} \|\tilde{u}_k\| + \|L\| \limsup_{k\to\infty} \left(\|\tilde{P}_k\| \|\bar{u}_k\| + \|\tilde{r}_k\| + \|d_k\| \right),$$

which by the convergence assumptions on P, r and d implies

$$\limsup_{k\to\infty} \|\tilde{u}_k\| \le \gamma \limsup_{k\to\infty} \|\tilde{u}_k\|.$$

The fact that the norm is nonnegative and $\gamma \in [0, 1)$ necessitates $\limsup_{k\to\infty} \|\tilde{u}_k\| = 0$. Thus, u converges to $\bar{u}$. Similarly, by (5.14) we have

$$\limsup_{k\to\infty} \|\tilde{e}_k\| \le \left(\|\bar{P}\| + \rho\right) \limsup_{k\to\infty} \|\tilde{u}_k\| + \limsup_{k\to\infty} \left(\|\tilde{P}_k\| \|\bar{u}\| + \|\tilde{r}_k\| + \|d_k\| \right).$$

The convergence assumptions on the uncertain terms imply that the right hand side of the inequality tends to 0. Thus, e converges to $\bar{e}$, completing the proof. ■

Theorems 5.2 and 5.3 are significant results for the following reasons: First, the bounds in (5.11) and (5.12) are continuous increasing functions of the uncertainties quantified by the scalars ρ, ζ, and the disturbance magnitude $\|d\|$. As such, decreased levels of uncertainty imply that system response can be guaranteed to be closer to its nominal counterpart. Moreover, in the case where $\rho = \zeta = 0$ and $d = 0$, (5.11) and (5.12) predict that the asymptotic response is equal to that of the nominal system, as expected. Second, in the case that the uncertainties vanish asymptotically, we can guarantee that the nominal response can be recovered in the limit.

5.4.3 Design Trade-offs

As in the iteration invariant case, it is trivial to show that γ is a measure of the convergence speed[21] of the algorithm: Recall from Section 5.1 that the input and error converge to the forced response of the ILC system. Furthermore, we saw in Section 5.3 that the effect of the initial input vanishes geometrically with rate γ. Hence, lower values of γ correspond to faster convergence to the forced response of the system, and vice versa.

Let $\alpha \triangleq \|L\|/(1-\gamma)$. We note that from (5.6), the bound $\|\bar{u}_\infty\| \leq \alpha\|r\|$ can be derived for the nominal case. Plugging this into (5.11) and (5.12), without loss of generality, it is easy to see that both the input and output asymptotic errors ($\limsup_{k\to\infty} \|\tilde{u}_k\|$ and $\limsup_{k\to\infty} \|\tilde{e}_k\|$) decrease as α decreases. Moreover

$$\lim_{\alpha\to 0}\left(\limsup_{k\to\infty} \|\tilde{u}_k\|\right) = 0,$$

and

$$\limsup_{\alpha\to 0}\left(\limsup_{k\to\infty} \|\tilde{e}_k\|\right) \leq \zeta + \|d\|, \tag{5.15}$$

since $\lim_{\alpha\to 0} \|\bar{u}_\infty\| = 0$ by (5.6). However, we note that decreasing α might come at

[21] More strictly, the convergence speed of the algorithm would be the smallest γ satisfying (5.3).

the expense of steady state performance. In the simulation examples and experimental implementation, we will use this fact to design optimal algorithms given steady state performance constraints.

5.4.4 Constrained Optimal Design for Predictable Performance

By definition of α, the ILC problem can be formulated as a constrained minimization of the following form:

$$\begin{aligned} \underset{\substack{Q\in B(U,U)\\ L\in B(Y,U)}}{\text{minimize}} \quad & \frac{\|L\|}{1-\bar{\gamma}} \\ \text{subject to} \quad & \bar{\gamma} = \|Q - L\bar{P}\| + \rho\|L\| \le \sigma < 1, \\ & \|I - \bar{P}(I - Q + L\bar{P})^{-1}L\| \le \beta. \end{aligned} \tag{5.16}$$

In (5.16), the constraint $\|Q - L\bar{P}\| + \rho\|L\| < 1$ is the robust stability criterion, derived by applying the triangle inequality on the uncertainty set $\mathcal{P}$ described by Assumption 5.1. This constraint can be relaxed as

$$\sup_{H\in\mathcal{P}} \|Q - LH\| \le \zeta < 1,$$

at the expense of computational complexity. On the other hand, the constraint $\|I - \bar{P}(I - Q + L\bar{P})^{-1}L\| \le \beta$ sets a limit on the allowable nominal steady state error $\bar{e}_\infty$ since

$$\bar{y}_\infty = \bar{P}(I - (Q - L\bar{P}))^{-1}L\bar{r}, \tag{5.17}$$

and therefore

$$\bar{e}_\infty = (I - \bar{P}(I - Q + L\bar{P})^{-1}L)\bar{r}.$$

Thus, the objective of the nonlinear program (5.16) is to find a robust linear ILC update law with guaranteed nominal steady state performance, that minimizes the deviations from the nominal system. The program (5.16) will be solved numerically via the MATLAB command `fmincon` and verified via simulations and experiments in the following sections.

5.5 Description of the Experimental Setup

This section describes the experimental setup that will be used to verify the findings of the previous sections. We will be working with discrete time linear dynamic systems over a fixed finite horizon, i.e. the spaces $U = Y = \mathbb{R}^n$ for some positive integer n, equipped with the 2 norm. The plant set $\mathcal{P}$ is a bounded set of $n \times n$ lower triangular (causal) nonsingular matrices. Similarly, the learning operators Q and L are $n \times n$ real matrices. Here, the inherent delay of the plant is ignored by shifting the output [9]. For example, if the system has relative degree 1, we consider the matrix equation $y_k = \bar{P} u_k$, where

$$\begin{aligned} u_k &\triangleq \begin{bmatrix} u_k(0) & u_k(1) & \dots & u_k(n-1) \end{bmatrix}^T, \\ y_k &\triangleq \begin{bmatrix} y_k(1) & y_k(2) & \dots & y_k(n) \end{bmatrix}^T. \end{aligned} \tag{5.18}$$

Similarly, the reference vector is given as

$$r_k \triangleq \begin{bmatrix} r_k(1) & r_k(2) & \dots & r_k(n) \end{bmatrix}^T. \tag{5.19}$$

5.5.1 Plant Description

The experimental setup considered in our work is an Aerotech ALS 25010, a low profile high accuracy linear motion stage, controlled through dSPACE. The specifications of

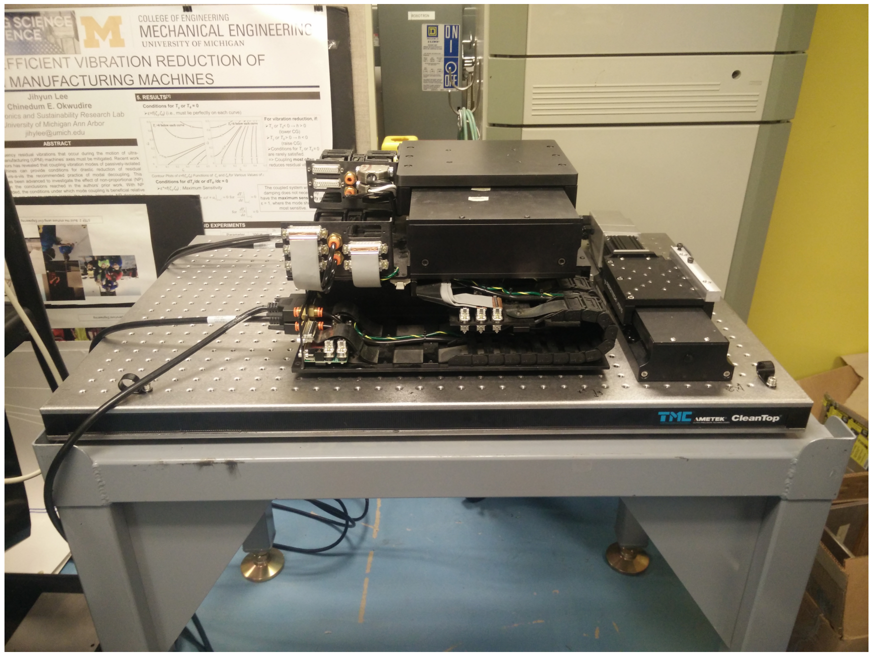


Figure 5.2: The experimental setup.

Table 5.1: Specifications of Aerotech ALS 25010[22]

Total Travel	100 mm
Servomotor	Brushless Linear
Encoder	Noncontact Linear
Resolution	0.001-0.2 μm
Maximum Travel Speed	2 m/s
Maximum Linear Acceleration	30 m/s^2
Accuracy	± 1 μm

the stage (the Y stage) are detailed in Table 5.1. The stage is mounted onto a similar Aerotech stage (the X stage), which in turn is connected to a 600×900 mm TMC breadboard. The motion ranges of the two stages are orthogonal to each other in Cartesian coordinates, thereby forming a dual axis XY type motion control platform. For simplicity, the latter of the stages is stabilized at a fixed position by a proportional-integral-derivative (PID) controller, and the overall setup is treated as a single axis motion stage.

[22] The travel speed and linear acceleration are limited to 300 mm/s and 3 m/s^2, respectively, by the software.

5.5.2 Closed Loop Identification

The Y stage is controlled by a PID controller (implemented at 1 kHz), resulting in the closed loop complementary sensitivity function

$$T_{\mathrm{cl}}(s) = KK_{\mathrm{p}}(K_{\mathrm{d}}s^2 + s + K_{\mathrm{i}}) \\ \times \frac{1}{Ms^3 + (C + KK_{\mathrm{p}}K_{\mathrm{d}})s^2 + (D + KK_{\mathrm{p}})s + KK_{\mathrm{p}}K_{\mathrm{i}}},$$

where the controller has proportional gain $K_{\mathrm{p}} = 5$, integral gain $K_{\mathrm{i}} = 0.3$, and derivative gain $K_{\mathrm{d}} = 3.51 \times 10^{-3}$. The function $T_{\mathrm{cl}}(s)$ is derived by combining the PID controller and the open loop empirically identified second order model with mass $M = 1\,\mathrm{kg}$, damping coefficient $C = 55\,\mathrm{Ns/m}$, spring coefficient $D = 2.6\,\mathrm{N/m}$, and open loop gain $K = 6660$.

It is well known that arbitrary small open loop modeling errors can lead to arbitrarily large closed loop modeling errors [63]. The identified closed loop model $T_{\mathrm{cl}}(s)$ is inaccurate for our purposes since ILC requires a relatively high bandwidth[23]. As such, a closed loop identification experiment at 1 kHz is performed in order to have an accurate impulse response of the closed loop, which can be used to construct the lower triangular Toeplitz plant matrix $\bar{P}$. This is done by sending a Heaviside step signal as the desired reference and differentiating the output signal. The first 200 samples of the identified impulse response are shown in Figure 5.3, where the signal is compared to the response $T_{\mathrm{cl}}^{\mathrm{disc}}(z)$ derived by discretizing $T_{\mathrm{cl}}(s)$ at 1 kHz.

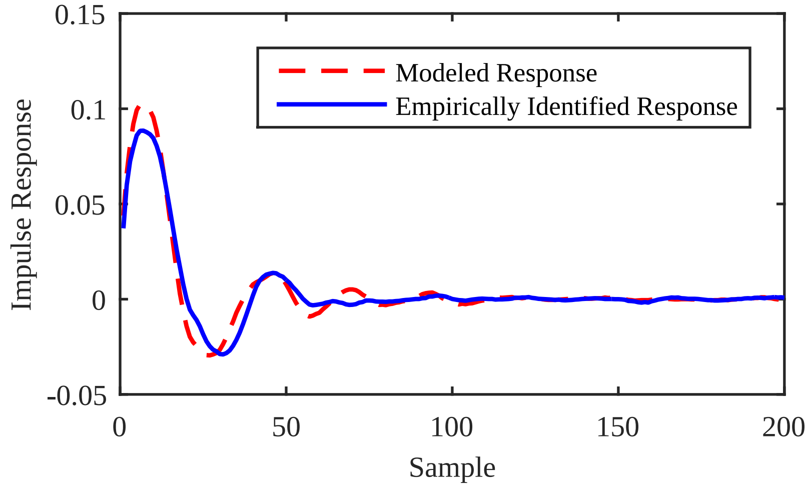


Figure 5.3: Modeled (via $T_{\text{cl}}^{\text{disc}}(z)$) and empirically identified closed loop impulse responses.

5.5.3 The Desired Output

The used reference signal is shown in Figure 5.4. It is a smooth ramp up and down signal at 1 kHz and lasts for 1 s. The signal covers approximately 75 percent of the Y-stage range and sets the velocity close to the software limit so that the reference is as challenging as possible, without excessive acceleration and jerk; this is done to avoid oscillations of the base that carries the breadbord and hence uncontrollable perturbations.

5.5.4 Plant Perturbations

Several weights varying between 100 g and 1.5 kg are used to perturb the experimental setup: During the experiments, these weights are placed on the Y stage according to a predetermined sequence $\mathcal{S}$ that was randomly chosen. As a result of the increased mass, the closed loop impulse response is perturbed. The magnitude of the pertur-

[23]For linear discrete time ILC, the relative degree, and the sign of the first nonzero Markov parameter is all that is needed for a stable update law. Similarly, for linear continuous time ILC, the relative degree and the sign of the corresponding feedthrough term is necessary and sufficient for a stable update law. However, the variety of algorithms that can be used with such limited information is small, and may result in slower convergence.

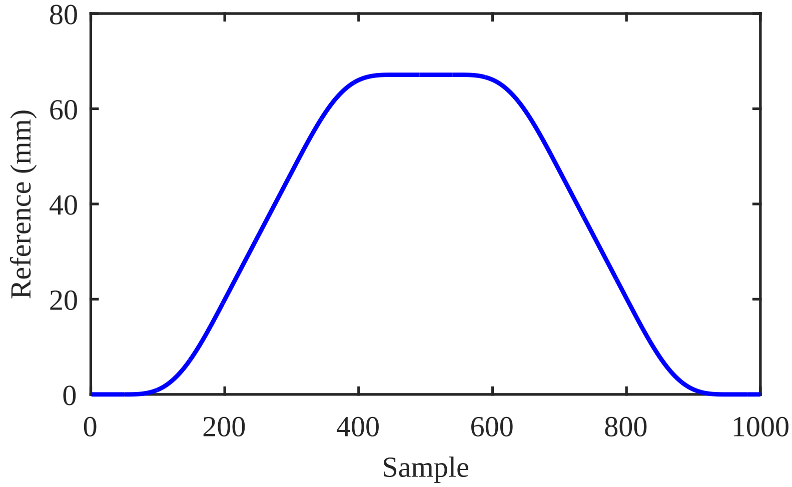


Figure 5.4: The desired output.

bations are roughly estimated to be $\rho = 0.01$ in terms of the uncertainty description of Assumption 5.1.

5.6 Simulations

As stated in Section 5.5, we will be working with the spaces $U = Y = \mathbb{R}^n$ for some positive integer n, equipped with the 2 norm. The plant set $\mathcal{P}$ is composed of $n \times n$ lower triangular nonsingular matrices, and the nominal plant $\bar{P}$ is derived from the closed loop identified impulse response (solid blue line) shown in Figure 5.3, unless otherwise stated. The objective of this section is twofold. First, the input-output stability of several well-known ILC algorithms under iteration varying uncertainties will be verified via simulation. Second, for certain classes of update laws, we will attempt to minimize the bounds on $\limsup_{k\to\infty} \|\tilde{e}_k\|$ using the nonlinear program (5.16) to obtain more predictable performance.

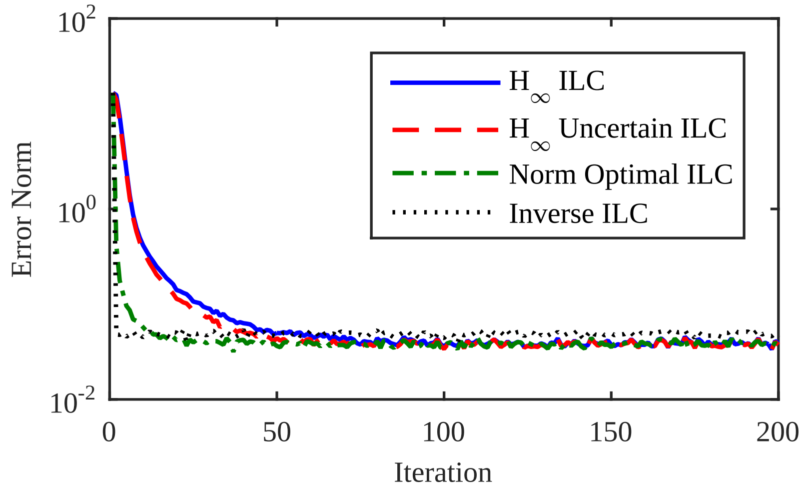


Figure 5.5: Comparison of several first and higher order algorithms under random perturbations. All algorithms maintain stability and boundedness under iteration varying disturbances and uncertainties. The higher order $\mathcal{H}_\infty$ ILC algorithms exhibit significantly slower convergence compared to the first order algorithms. While inverse ILC converges in a single iteration, it has a higher steady state error, since it is sensitive to plant uncertainties and disturbances.

5.6.1 Stability under Iteration Varying Perturbations

Figure 5.5 compares the performance of four different ILC algorithms in the presence of trial varying uncertainties and disturbances. The additive plant uncertainty $(P_k - \bar{P})$ is chosen to be a lower triangular random matrix, where each nonzero entry is drawn from $\mathcal{U}(-0.005, 0.005)$. Similarly, disturbances are considered to be a combination of input and output disturbances $d_k^{\text{in}}, d_k^{\text{out}}$, where each entry is drawn from $\mathcal{U}(-0.0025, 0.0025)$.

The ILC algorithms considered in this scenario are listed as follows:

1. $\mathcal{H}_\infty$ ILC for certain systems.

2. $\mathcal{H}_\infty$ ILC for uncertain systems.

3. Norm optimal ILC, in which the quadratic cost function J is minimized by

solving for u_{k+1} without constraints;

$$J = e_{k+1}^T W_e e_{k+1} + u_{k+1}^T W_u u_{k+1} + (u_{k+1} - u_k)^T W_{\Delta u}(u_{k+1} - u_k), \quad (5.20)$$

where $W_e, W_u, W_{\Delta u}$ are positive (semi) definite matrices. To simplify the problem further for the norm optimal framework (5.20), we will assume that these weighting matrices are scalar multiples of the identity matrix, so $W_e = w_e$, $W_u = w_u$, $W_{\Delta u} = w_{\Delta u}$. The algorithm in Figure 5.5 is derived by setting the weighting parameters as $w_e = 1$, $w_u = 0$, and $w_{\Delta u} = 0.5$, which are heuristically tuned.

4. Inverse ILC, i.e. $Q = I$ and $L = \bar{P}^{-1}$. Note that the matrix $\bar{P}$ is invertible since the plant set $\mathcal{P}$ comprises nonsingular matrices.

The $\mathcal{H}_\infty$ type ILC algorithms are described in detail in [12] and in general yield higher order (up to order n for n samples) algorithms. However, it is a straightforward exercise to extend our analysis to nth order algorithms by augmenting (5.1); e.g. we can consider $y_k^{\text{aug}} = (y_k, y_{k+1}, \ldots, y_{k+n-1})$. The reader can see in Figure 5.5 that all algorithms maintain stability and boundedness under iteration varying disturbances and uncertainties. It is also worth noting that the higher order $\mathcal{H}_\infty$ ILC algorithms exhibit significantly slower convergence compared to the first order algorithms.

5.6.2 Computation and Verification of Optimal Update Laws

To demonstrate the utility of the optimization approach to ILC design, the performance of different Q and L matrices computed via (5.16) will be compared. For each of the computed algorithms, a set of 200 trials will be conducted, and for each algorithm there exist positive integers N_0 and N_{f} such that disturbances and uncertainties affect the system from trial N_0 to N_{f}. The performance measure we would like to minimize is given as

$$\delta \triangleq \max_{k,j\in\{N_0,N_0+1,\ldots,N_{\mathrm{f}}\}} \|e_k - e_j\|. \tag{5.21}$$

The scalar quantity δ is an indirect measure of fluctuations from nominal performance, with lower values signifying better predictability with respect to the nominal system. The reason for considering this measure as opposed to $\max_{k\in\mathbb{N}} \|\tilde{e}_k\|$ is consistency with the experimental validation, since the "nominal" system is not implementable in practice due to noise and disturbances.

The following steps are taken to enhance computational aspects of the problem:

- **Norm optimally derived filters:** The first case we consider is that the update law is derived via the norm optimal framework (5.20) with scalar weighting matrices, so

  $$J = w_e\|e_{k+1}\|^2 + w_u\|u_{k+1}\|^2 + w_{\Delta u}\|u_{k+1} - u_k\|^2. \tag{5.22}$$

  The solution of the norm optimal problem is given in the form of matrices Q, L such that

  $$u_{k+1} = Qu_k + Le_k. \tag{5.23}$$

  In other words, we impose the additional constraint on (5.16) that the matrices Q, L minimize the cost function (5.22) via (5.23).

  A specific solution (Q, L) for a given nonzero $(w_e, w_u, w_{\Delta u})$ is invariant over the set $\{\mu(w_e, w_u, w_{\Delta u}) : \mu \in (0, \infty)\}$. As such, the weighting w_e can be fixed so that the program (5.16) with the additional constraint defined above optimizes over the two scalars w_u and $w_{\Delta u}$.

- **Lower triangular Toeplitz filters:** In a similar fashion, to reduce complexity, we will also consider the case where Q and L are lower triangular Toeplitz matrices. This reduces the number of variables to be optimized from $2n^2$ to $2n$, significantly decreasing the computational burden. Despite this simplification, the program (5.16) is still computationally expensive for large n. For demonstration purposes, the number of samples for this simplification will be chosen as 10, and the model used will be the discretization of the identified closed loop model $T_{\mathrm{cl}}(s)$ sampled at 100 Hz. The considered reference signal is a 5 Hz unit amplitude sine wave. Note that since the output and the reference are shifted via (5.18) and (5.19), the matrix L represents a noncausal LTI filter when it is nonsingular: The input $u_k(i)$ at time i depends on the error $e_k(i+1)$, for all $i \in \{0, 1, \ldots, n-1\}$.

We also note that similar simplifications can be made, for example, by choosing Q and L to be diagonal, or upper triangular and/or Toeplitz. As before, the additive plant uncertainties will be chosen to be lower triangular random matrices, where each nonzero entry is drawn from $\mathcal{U}(-0.005, 0.005)$. Similarly, the disturbances are considered to be a combination of input and output disturbances $d_k^{\mathrm{in}}, d_k^{\mathrm{out}}$, where each entry is drawn from $\mathcal{U}(-0.0025, 0.0025)$.

Remark 5.1. At first glance, optimizing an "optimal" learning law might seem redundant, but can be explained by analogy to linear quadratic regulation (LQR). LQR is an optimal control methodology in which a quadratic "cost" function is minimized to find an optimal state feedback law. In practice, the cost function and the associated weighting matrices are not given as the design specification for a control problem. Often, the weighting matrices are used as "tuning knobs" to properly adjust the resulting state feedback law and achieve given design specifications (e.g. maximum rise time and/or settling time, minimum disturbance rejection etc.). In this sense, our approach is similar to optimally selecting the LQR weights to minimize plant sensitivity, subject to a lower bound on convergence rate and an upper bound on steady state error under step responses, which can be done in a numerical fashion.

Table 5.2: Simulation comparison of optimized update laws derived from (5.22) for different values of β. Here, $w_e = 10$ is the fixed weighting so the optimization is over w_u and $w_{\Delta u}$. The additive plant uncertainties are chosen to be lower triangular, where each nonzero entry is drawn from $\mathcal{U}(-0.005, 0.005)$. Similarly, each entry of $d_k^{\text{in}}, d_k^{\text{out}}$ is drawn from $\mathcal{U}(-0.0025, 0.0025)$. For each case, $\|\bar{e}_\infty\| = \beta$.

β	w_u	$w_{\Delta u}$	$\bar{\gamma}$	α	δ
0.9500	0.0263	0	0.0975	10.803	183.05
0.9990	1.1383	0	0.0134	1.3623	129.95
0.9999	10.000	0	0.0050	0.5025	70.29

Table 5.2 compares update laws derived from (5.22) for different values of β, which bounds the acceptable steady state error level. For all cases, the nominal asymptotic error turns out to have magnitude β; i.e. $\|\bar{e}_\infty\| = \beta$. It can be seen that decreasing values of α signify a decreasing level of performance uncertainty, i.e. decreasing δ. Moreover, there seems to be a trade-off between β and δ, so predictable performance comes at the expense of nominal performance.

The norm optimal framework (5.22) gives limited design freedom since only two scalar variables are optimized. The usefulness of the optimization approach (5.16) can be seen better in Figure 5.6, where 10×10 lower triangular Toeplitz matrices Q and L are optimized, as noted before. To further verify the trade-off between α and δ, different lower bounds on α are set as optimization constraints, while β is kept constant. The update law with $\alpha = 0.6882$ yields more predictable performance compared to when $\alpha = 3.3345$, which can also roughly be seen from the fact that the latter achieves a higher maximal and and a lower minimal error, while the nominal asymptotic performance is the same.

5.7 Experimental Results

In this section, experimental implementation results for the update laws derived in Section 5.6 will be presented. The objectives of the section are similar to that of Section 5.6. That is, we would like to verify experimentally, the input-output stability

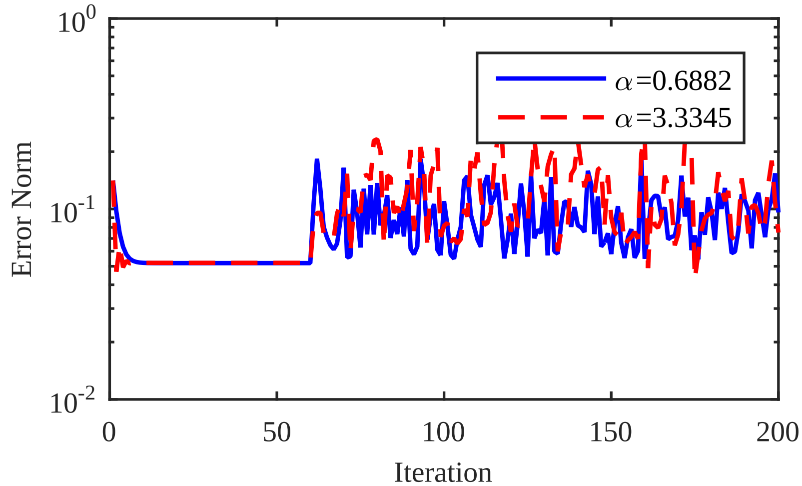


Figure 5.6: Performance of optimized lower triangular Toeplitz controllers with: For $\alpha = 3.3345$ we have $\delta = 8.3930$, and for $\alpha = 0.6882$ we have $\delta = 3.1896$. The additive plant uncertainties are chosen to be lower triangular, where each nonzero entry is drawn from $\mathcal{U}(-0.05, 0.05)$. Similarly, each entry of $d_k^{\text{in}}, d_k^{\text{out}}$ is drawn from $\mathcal{U}(-0.25, 0.25)$.

of a couple of ILC algorithms under iteration varying uncertainties. Second, we would like to roughly verify the optimization approach (via the nonlinear program (5.16)) to norm optimal ILC synthesis by comparing the experimental performance of the update laws whose simulation results are shown in Table 5.2. As an additional point, we will discuss the idea of *precompensation in the iteration domain* and test this idea on our experimental setup.

5.7.1 Robust Stability of First and Higher Order ILC

We will compare the $\mathcal{H}_\infty$ ILC algorithm for certain systems described in [12] with a simple manually tuned norm optimal controller; the particular $\mathcal{H}_\infty$ algorithm is chosen since it requires significantly less time to be synthesized and has similar performance compared to its uncertain counterpart (see Figure 5.5). For robustness against high frequency noise buildup, the computed input u_{k+1} of the $\mathcal{H}_\infty$ controller is further filtered through a first order low pass filter with a cutoff frequency of 400 Hz. The norm optimal controller has the scalar weightings $w_e = 10$, $w_u = 0$, and $w_{\Delta u} = 5$. At the samples where the velocity of the reference signal is equal to 0, a first order low pass filter with cutoff frequency of 150 Hz is applied to ensure robustness against high frequency noise buildup and avoid numerical instability. The results can be seen in Figure 5.7, where both systems maintain stability and portray comparable performance under unknown bounded perturbations from trials 25 to 45, where the predefined sequence $\mathcal{S}$ of weights are placed on the Y stage.

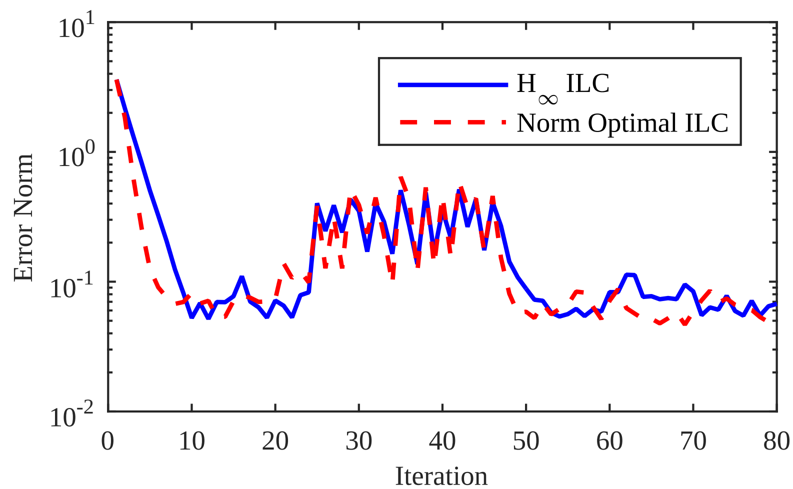


Figure 5.7: Comparison of $\mathcal{H}_\infty$ and norm optimal ILC algorithms under bounded unknown perturbations from trials 25 to 45, where the predefined sequence $\mathcal{S}$ of weights are placed on the Y stage from trials 25 to 45. Both systems maintain stability and portray comparable performance under unknown bounded perturbations.

5.7.2 Optimized Update Laws

The norm optimal controllers derived from (5.16), whose simulation results are shown in Table 5.2, are tested on the experimental setup to verify the hypothesis that δ can be minimized via the program (5.16). However, to avoid high frequency noise buildup, we fix $w_{\Delta u} = 1$. The predefined sequence $\mathcal{S}$ of weights are placed on the Y stage as before from trials 25 to 45. We note that we use the scalar quantity δ defined in (5.21), since the "nominal" system is not implementable in practice due to noise and disturbances. As can be seen in Table 5.3, decreasing values of α signify

Table 5.3: Experimental comparison of optimized update laws derived from (5.22) for different values of β. Here, $w_e = 10$ is the fixed weighting so the optimization is over w_u and $w_{\Delta u}$. For each case, $\|\bar{e}_\infty\| = \beta$. Decreasing values of α signify decreasing values of δ, as expected.

β	w_u	$w_{\Delta u}$	$\bar{\gamma}$	α	δ
0.9500	0.0263	1	0.0975	57.9360	1.2274
0.9990	1.1383	1	0.4193	1.7637	1.1830
0.9999	10.000	1	0.0909	0.9689	0.5244

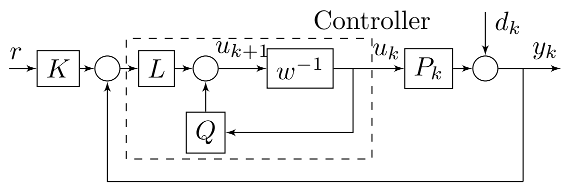


Figure 5.8: Precompensation in the iteration domain: The feedback control in the iteration domain interpretation of ILC makes it clear that aggressive learning might amplify disturbances. When the controller does not have an integrator, i.e. $Q \neq I$, the precompensator K can be used to partially recover the tracking performance.

decreasing values of δ, which is expected. Note that δ values are much lower compared to their simulated values, which is due to the fact that the experimental perturbations are limited to several different weights as opposed to the random perturbations of the simulation scenarios.

5.7.3 Precompensation in the Iteration Domain

Perfect tracking is an infeasible objective when the system to be controlled is subject to unknown iteration varying disturbances and/or, when the additive uncertainty is high in magnitude. As such, depending on the magnitude of uncertainties, minimizing the measure φ can be taken as an objective of primary importance over the steady state performance. This approach has not been explored much in the ILC literature. To be precise, while plenty of publications have studied how to reduce the absolute error, not much work has been done to quantify the *relative error* $\tilde{e}_k$ in the presence of iteration varying effects. For certain applications (e.g. manufacturing), precision is

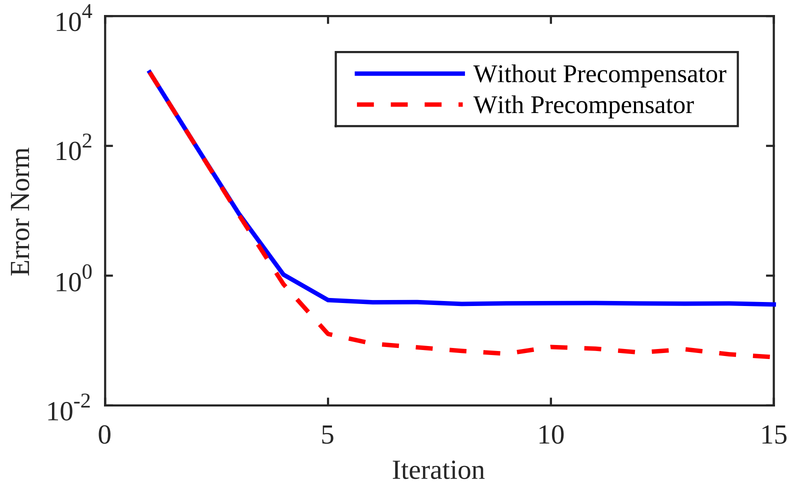


Figure 5.9: Experimental verification of precompensated norm optimal ILC, with $w_e = 10$, $w_u = 0.0025$, and $w_{\Delta u} = 1$. Precompensation leads to an order of magnitude decrease in the norm of the error $r - y_k$ as $k \to \infty$.

arguably more important than accuracy, and repeatable errors are preferred. When this is the case and perfect tracking is infeasible or undesirable due to large uncertainties, and/or iteration varying effects, we propose *precompensation in the iteration domain* (see Figures 5.8) as an ad hoc fix to recover tracking performance: Pole placement methods typically change DC gains of systems, which are commonly recovered through precompensation, and this idea can be easily extended to ILC systems. One simple choice for the precompensator K is given by inverting the nominal steady state reference to output matrix given in (5.17),

$$K = (\bar{P}(I - (Q - L\bar{P}))^{-1}L)^{-1},$$

which is verified experimentally: Figure 5.9 shows that precompensation results in approximately an order of magnitude improvement in tracking, i.e. an order of magnitude decrease in the norm of the error $r - y_k$. Moreover, the precompensated system maintains stability in the presence of perturbations, as can be seen in Figure 5.10.

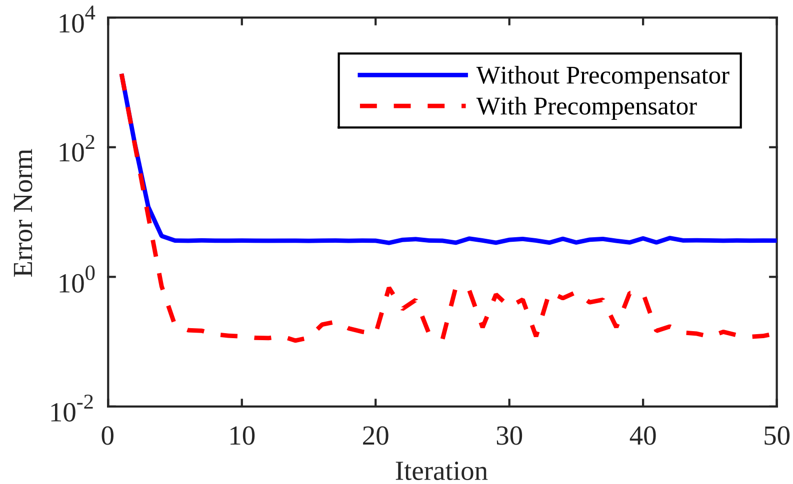


Figure 5.10: Experimental verification of precompensated norm optimal ILC subject to mass perturbation sequence $\mathcal{S}$ with $w_e = 10$, $w_u = 0.0025$, and $w_{\Delta u} = 1$. Precompensation leads to more than an order of magnitude decrease in the norm of the the error $r - y_k$ as $k \to \infty$, and does not affect robust stability.

5.8 Conclusion

In this chapter, we scrutinized the stability and convergence properties of ILC systems subject to trial to trial uncertainty. We formulated the system to be controlled as a linear input-output map in an abstract Banach space setting to ensure the generality of our analysis, assuming bounded uncertainties in all process parameters; including the input-output operator, the feedback response, reference, noise, disturbance and initial conditions. We showed that when a linear update law is designed to be robust over the set of possible maps $\mathcal{P}$, linear discrete time methods can be employed directly to show the system exhibits desirable properties such as asymptotic stability and boundedness. Moreover, we investigated how the design of the operators Q and L affects the convergence properties of iteration varying systems. We showed that an iteration varying system converges to 1) a bounded neighborhood of a nominal system if the uncertainties are bounded, and, 2) the nominal system itself if the uncertainties are convergent. Further we argued for employing an optimization based approach to ILC design to improve predictability in iteration varying systems. Our analysis was supported by simulation results, along with experimental verification on a linear motion control stage.

It turns out that robust ILC methods, which are well studied in the literature, can be applied directly to iteration varying systems. The results are quite strong in terms of their generality and the lack of limiting assumptions apart from linearity. A further direction to pursue is the study of optimal ILC strategies with structured (time invariant, higher order etc.) perturbations under discrete or continuous frameworks, with or without feedback. A disturbance rejection problem has been considered in [123] via l_1 norm minimization, and an $\mathcal{H}_\infty$ minimization problem for HOIM based plants, references, disturbances has recently been considered in [121]. We expect the initial results of this chapter, along with some of the work in [121, 123] to pave the way for future research in iteration varying systems in ILC.

CHAPTER 6

From ILC to 2D Systems: Exponential Stability of Nonlinear Differential Repetitive Processes

In this chapter, we will extend the notion of learning through repeated trials to 2D dynamic systems, and take the first steps towards synthesizing multidimensional control algorithms for a variety of applications, by deriving necessary and sufficient exponential stability conditions for *repetitive processes*. The development of a stability theory for these classes of systems will help in the generalization of the learning paradigm for a plethora of applications, along the feedback in the iteration domain interpretation of ILC.

Recall from the earlier chapters that repetitive processes are systems where the dynamics at trial k is affected by the output history $y_0, y_1, \ldots, y_k$. Specifically, repetitive processes are 2D dynamic systems that arise in the modeling of engineering applications, in which information propagation occurs along two axes of independent variables. These processes are characterized by a sequence of passes with *finite length* that act as forcing functions on the dynamics of future passes [2], hence the name multipass. On an abstract level, recursive algorithms for one dimensional (1D) dynamic systems can be treated as repetitive processes; e.g. iterative solutions to nonlinear optimal control problems [2, 124, 125], iterative nonlinear inversion [23], iterative estimation and control design [63], or the constructive proof of the Picard-Lindelöf theorem. In the application space, typical examples to these classes of systems include long wall coal cutting [126, 127], metal rolling [128, 129], or AM systems such as LMD [29, 30, 130]. We also note that as we have mentioned before, ILC can be thought of as a special class of repetitive processes [2, 12], wherein the pass to pass dynamics are induced through the construction of a recurrence relation that updates the feedforward input using past data. The application of 2D systems theory based ILC synthesis can be found as early as 1990s [67, 131, 132], while repetitive process based ILC laws have been experimentally verified in recent years [2, 133, 134].

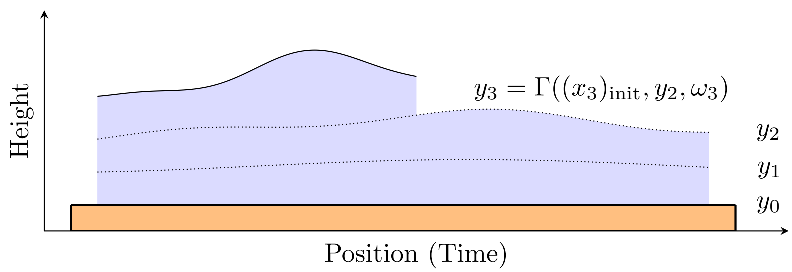


Figure 6.1: AM systems as repetitive processes: The substrate topography determines the initial output y_0. The operator Γ maps the initial state $(x_3)_{\text{init}}$ and input ω_3 of pass 3 (in layer dynamics), along with the prior pass profile y_2 (layer to layer dynamics), to pass profile y_3. The layer to layer dynamics is affected by physical phenomena such as heat transfer and material curing.

An example to the repetitive process paradigm is the generic AM system demonstrated in Figure 6.1. In this setup, the material profile at pass 3 is determined by the operator Γ comprising the in layer and layer to layer dynamics. In the ideal case, the layer to layer dynamics would be a discrete integrator along the pass domain. A more concrete example can be found in [29], wherein a height dependent model taking heat transfer from prior layers into account is developed, and in [130], which presents a repetitive process control strategy through a control oriented model, for the LMD process. A less rigorous treatment of the repetitive process interpretation of AM is given in [135], which models edge shrinking in ink jet printing through a 2D discrete convolution like formula, albeit in three dimensions.

The control oriented modeling and linear repetitive process control approach to the LMD process in [130] is the primary motivator of this chapter: The developed model in [130] consists of static nonlinearities, while the controlled process is assumed to be locally stable around its *linearized* equilibrium. Indeed, nonlinear analysis of these systems has not been studied in much detail, and current literature on repetitive processes and 2D systems theory is predominantly on linear stability and performance analysis, and control synthesis (see [2, 136, 137] and references therein). On the other hand, the need to develop rigorous stability tests in the nonlinear systems context has been highlighted only very recently. Among these works, [138] presents Lyapunov theorems for nonlinear Roesser models evolving in the domain $\mathbb{N}^2$, with extensions to the stochastic case given in [139], and a 2D Lyapunov function approach is employed to prove exponential stability of systems described by a differential repetitive process (DRP) in [140]. As a secondary motivation, in the ILC literature, it has been noted that nonlinear update laws have not been extensively researched, save for adaptive laws for locally Lipschitz plants, and a systematic theory of nonlinear ILC is an open question [8, 31, 33].

With these issues in mind, in this chapter, we establish a DRP analogue of the well known result that exponential stability of a nonlinear 1D feedback system is equivalent to the exponential stability of the linearized dynamics. The results of this chapter are an extension of our prior work in [41], in which we show that exponential stability of a time invariant DRP can be verified by the stability of its linearization, along with two small gain conditions. With respect to [41], our contribution can be summarized as follows:

1. The problem statement is relaxed to allow for time varying processes; linear stability theory is extended to the linear time varying (LTV) case.
2. The stability requirement in the time domain is bypassed by adopting a small signal existence, uniqueness, and boundedness condition.

3. The exponential stability definition is modified to be more in line with its 1D counterpart.

4. The finite time aspect of the problem is utilized to remove the $\mathcal{H}_\infty/\mathcal{L}_1$ small gain conditions.

5. 2D Lyapunov equation based theorems are interchanged with abstract Lyapunov theorems on the function space, which are developed by treating the model as a discrete system on a Banach space.

The Lyapunov theorems mentioned above are then used to show that *a DRP is exponentially stable if and only if its corresponding linearization is stable*, an analogue of the well known result that the exponential stability of a nonlinear one dimensional (1D) system is equivalent to the exponential stability of the linearized dynamics. Moreover, we show that the exponential stability of a linear DRP is equivalent to the spectral radius of the state matrix $\bar{D}$ being uniformly less than 1 over the interval $[0, T]$, strengthening the findings of [41].

The rest of the chapter is organized as follows: Section 6.1 gives the necessary background and introduces the style of notation to be used. Section 6.2 introduces state space representations of DRPs, establishes the key Lipschitz property of the nonlinear operator, and states formal stability definitions. In Section 6.3, we develop Lyapunov like theorems for these classes of systems. Stability theory for linear systems is extended to the time varying case in Section 6.4. Our main result, which establishes equivalence in terms of exponential stability between a DRP and its linearization, is presented in Section 6.5. Applications of this result to Picard iterations and ILC is discussed in Section 6.6. An illustrative example is given in Section 6.7 through an ILC system. Concluding remarks are given in Section 6.8. In the hope of improving readability, proofs of certain technical results are given in Appendices B.1, B.2, and B.3.

6.1 Background and Preliminaries

We start by giving a concise background on linear repetitive processes and introducing our style of notation for clarity of exposition. Two basic inequalities that will be used in the rest of this chapter are also proven.

6.1.1 Notation

We use $\mathbb{R}$ to represent real numbers, $\mathbb{R}^+$ its nonnegative subset, $\mathbb{N}$ nonnegative integers, and $\mathbb{C}$ complex numbers. The spectral radius of a linear operator is denoted by $\rho(.)$. The identity and zero operators are denoted as I and 0, respectively. For a real vector, $\|.\|_2$ is the 2 norm; in the rest of the chapter $\|.\|$ will denote any of the equivalent norms in the real space. $\mathcal{L}_p$ is the space of measurable functions on the compact interval $[0, T]$ with finite $\mathcal{L}_p$ norm, $p \in [1, \infty]$. We will also make use of c_0, the space of all real sequences of given size that converge to 0.

6.1.2 Linear Repetitive Processes in Banach Space

A general abstract model of a linear repetitive process assumes an underlying Banach space structure [2], and can be thought of as the discrete counterpart of the abstract inhomogeneous Cauchy problem in infinite dimensional systems [141]. In particular, we assume that the output at pass (or iteration) k, denoted y_k, is a vector in a closed subspace $\mathcal{Y}_T$ of a complete function space $\mathcal{Y}$, where $T < \infty$ denotes the duration or length of the pass profile. Then, $y_{k+1} = L_T y_k + \omega_{k+1}$ for all $k \in \mathbb{N}$, where $L_T \triangleq L|_{\mathcal{Y}_T}$ is the restriction[24] of the bounded linear operator L, and $\omega_k \in \mathcal{Y}_T$ is a vector that represents the effect of initial conditions, disturbance, noise, and the control input. In this work, we will study operators like L that are not necessarily linear, and are described by differential equations, hence *differential repetitive processes*. In the rest

[24]Here, we also restrict the codomain of L to Y_T via some truncation like operation.

of the chapter, we will drop the subscript T from $\mathcal{Y}_T$ for convenience.

6.1.3 Useful Inequalities

The inequalities below will be of use for convergence analysis; note that the exponential convergence parameters $2/(1-a) \geq 1$ and $(1+a)/2 \in (0,1)$ are continuous increasing functions of a on $(0,1)$.

Claim 6.1. *Let* $\mathbf{a} \triangleq \{a_{k+1}\}_{k=0}^{\infty}$ *and* $\mathbf{b} \triangleq \{b_{k+1}\}_{k=1}^{\infty}$ *be real nonnegative sequences, where* $\mathbf{b}$ *is bounded, such that* $a_{k+1} = ra_k + b_{k+1}$ *for some* $r \in (0,1)$ *for all* $k \in \mathbb{N}$. *Then,* $\limsup_{k\to\infty} a_k \leq (1/(1-r)) \limsup_{k\to\infty} b_k$, *and therefore* $\mathbf{b} \in c_0$ *implies* $\mathbf{a} \in c_0$.

Proof. Boundedness of $\mathbf{a}$ is readily verified as r is Schur. Taking limit superiors of both sides of the equality, $\limsup_{k\to\infty} a_k \leq r \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k$, since $r > 0$ and both sequences are positive. Rearranging this yields the desired result. ■

Claim 6.2. *Let* $a \in (0,1)$. *Then the sequence* $\{ka^{k-1}\}_{k=0}^{\infty}$ *is exponentially convergent and*

$$ka^{k-1} \leq \frac{2}{1-a}\left(\frac{1+a}{2}\right)^k, \quad \forall k \in \mathbb{N}.$$

Proof. Consider the alternative statement

$$\frac{2a}{1-a}\left(\frac{1+a}{2a}\right)^k - k \geq 0, \quad \forall k \in \mathbb{N},$$

which can be proven by induction: The inequality is true for $k = 0$. Then, one can show that the increase in the left side of the inequality from k to $k+1$ is nonnegative for every $k \in \mathbb{N}$, since $(1+a)/(2a) > 1$. ■

6.2 State Space Formulation of DRPs

Throughout this chapter we will study systems of the form

$$\begin{aligned} \dot{x}_{k+1}(t) &= f(x_{k+1}(t), y_k(t), t), \\ y_{k+1}(t) &= g(x_{k+1}(t), y_k(t), t), \end{aligned} \tag{6.1}$$

for all $t \in [0, T]$ and $k \in \mathbb{N}$, for some $T \in \mathbb{R}^+$. The vectors $x_k(t) \in \mathbb{R}^n$ and $y_k(t) \in \mathbb{R}^m$ represent the state and output, respectively. Note that it is also necessary to specify boundary conditions y_0 and $\mathbf{x}(0) \triangleq \{x_{k+1}(0)\}_{k=0}^{\infty}$ to uniquely determine the solution. In this model, we ignore any exogenous inputs since our goal is to study Lyapunov stability, whose definition will be made precise later.

By concatenating the passes (e.g. letting $x(\tau) \triangleq x_k(t)$ where $\tau = t + kT$), it is also easy to see that the model of (6.1) resembles hybrid systems with memory [142]: Here, T plays the role of delay, with the flow condition $t(\tau) \in [0, T]$ and the jump condition $t(\tau) \in \{T\}$, where the timer variable t is subject to the flow $\dot{t}(\tau) = 1$ and jump $t^+(\tau) = 0$, a la sample and hold control systems. Similarly, the index $k(\tau) \in \mathbb{N}$ is subject to $\dot{k}(\tau) = 0$ and $k^+(\tau) = k(\tau) + 1$. The main differences of this formulation as compared to hybrid systems with memory are, 1) the existence of the output equation $y(\tau) = g(x(\tau), y(\tau - T), t)$, 2) the fact that *the delay occurs via the output* as opposed to the state, in both the output and state equations, and 3) the *arbitrary*, time (iteration) varying nature of the jump equation $x^+(\tau) = x_{k+1}(0)$. To define stability, we will impose a specific structure on this equation, i.e. $\mathbf{x}(0)$, as we shall see in the following sections. See [2] for DRP modeling of delay differential equations.

6.2.1 The Nonlinear Operator over the Finite Horizon

Before proceeding with further analysis, we will look at the properties of the system (6.1) as an input-state and input-output operator over the time interval $[0, T]$:

Interchanging y_k with u, x_{k+1} with χ, and y_{k+1} with w, we consider

$$\begin{aligned} \dot{\chi}(t) &= f(\chi(t), u(t), t), \\ w(t) &= g(\chi(t), u(t), t), \end{aligned} \tag{6.2}$$

for all $t \in [0, T]$. The input u resides in $\mathcal{Y}$, the space of continuously differentiable functions on $[0, T]$. We will impose the following assumptions on the nonlinear system (6.2) that maps the pair $(\chi(0), u)$ to χ and w:

Assumption 6.1. The nonlinear system (6.2) satisfies the following conditions:

1. The functions f and g vanish at the origin uniformly in time. That is

$$f(0, 0, t) = 0, g(0, 0, t) = 0 \forall t \in [0, T].$$

2. There exists $\delta > 0$ such that for every $(\chi(0), u)$ satisfying $\|\chi(0)\| + \|u\|_{\mathcal{L}_\infty} < \delta$, there is a unique integral curve χ of (6.2), and $\chi(t)$ which is contained in a bounded open connected set X for all $t \in [0, T]$.

3. There exists a compact set $Y \subset \mathbb{R}^m$ that contains the origin in its interior such that f and g are continuously differentiable in $Z \triangleq \mathrm{cl}(X) \times Y \times [0, T]$, where $\mathrm{cl}(X)$ is the closure of X.

Assumption 6.1 is a relatively mild constraint on the system that bypasses the stability requirement in the time domain. One way of ensuring this is to enforce input to state stability over an infinite horizon, or the notion of finite time uniformly bounded energy bounded state [143] stability, so that the trajectories of the differential equation are within a compact set.

We note that since 0 is an equilibrium of the differential equation, the set X must contain the origin. Without loss of generality, we will also assume that δ is small enough so that $\chi(0) \in X$ and $u(t) \in Y$ for all $t \in [0, T]$ when $\|\chi(0)\| + \|u\|_{\mathcal{L}_\infty} < \delta$.

We denote by Γ_x the mapping $(u, \chi(0)) \mapsto \chi$, and by Γ_y the mapping $(u, \chi) \mapsto y$.Then the nonlinear operator Γ can be defined so that

$$(w, \chi) = \Gamma(u, \chi(0)) \triangleq (\Gamma_y(u, \Gamma_x(u, \chi(0))), \Gamma_x(u, \chi(0))).$$

Now, we can show Lipschitz continuity of the operator Γ.

Lemma 6.1. *The nonlinear differential equation Γ_x in* (6.2)*, is locally Lipschitz with respect to $(\chi(0), u)$. That is, there exists a constant L_1 such that if χ_i is the integral curve of* (6.2) *corresponding to $(\chi_i(0), u_i)$, for all $i \in \{1, 2\}$,*

$$\|\chi_1 - \chi_2\|_{\mathcal{L}_\infty} \leq L_1(\|u_1 - u_2\|_{\mathcal{L}_\infty} + \|\chi_1(0) - \chi_2(0)\|),$$

when $\|\chi_i(0)\| + \|u_i\|_{\mathcal{L}_\infty} < \delta$, for all $i \in \{1, 2\}$.

As an immediate corollary, we have the following:

Corollary 6.1. *The input-output operator* (6.2)*, $\pi_y \circ \Gamma$, where π_y is the standard projection onto $\mathcal{Y}$, is locally Lipschitz with respect to $(\chi(0), u)$. That is, there exist positive constants L_2 and $\bar{\delta}$ such that if w_i is the output of* (6.2) *corresponding to $(\chi_i(0), u_i)$, for all $i \in \{1, 2\}$,*

$$\|w_1 - w_2\|_{\mathcal{L}_\infty} \leq L_2(\|u_1 - u_2\|_{\mathcal{L}_\infty} + \|\chi_1(0) - \chi_2(0)\|)$$

when $\|\chi_i(0)\| + \|u_i\|_{\mathcal{L}_\infty} < \bar{\delta}$, for all $i \in \{1, 2\}$.

6.2.2 Boundary Dependent Stability Definitions

We will now lay out definitions of stability for DRPs. First, we need the following norm to characterize exponential initial state sequences for exponential stability.

Definition 6.1. Let $\mathbf{b} \triangleq \{b_{k+1}\}_{k=0}^{\infty}$ be a sequence of real vectors of given size. The exponential λ (e_λ) norm of $\mathbf{b}$ is defined as $\|\mathbf{b}\|_{e_\lambda} \triangleq \sup_{k\in\mathbb{N}} \lambda^{-k}\|b_{k+1}\|$, for all $\lambda \in (0, 1]$.

We leave it to the reader to verify that e_λ, the space of all sequences with finite e_λ norm, is precisely the vector space of sequences that converge exponentially to 0 with rate faster than or equal to $-\ln(\lambda)$; i.e. the geometric convergence factor has to be smaller than or equal to λ. The e_1 norm, on the other hand, is precisely the sup norm. Note that $e_\lambda \subset c_0 \subset e_1 \equiv l_\infty$, for all $\lambda \in (0, 1)$. Given any $\kappa \in \mathbb{N}$, the e_λ norm also has the property

$$\|\mathbf{b}_\kappa\|_{e_\lambda} \le \lambda^\kappa \|\mathbf{b}\|_{e_\lambda}, \tag{6.3}$$

where $\mathbf{b}_\kappa \triangleq \{b_{k+1}\}_{k=\kappa}^{\infty}$. In addition $\|.\|_{e_{\lambda_2}} \le \|.\|_{e_{\lambda_1}}$ when $0 < \lambda_1 \le \lambda_2 \le 1$.

Definition 6.2. The (origin of the) DRP (6.1) is said to be

1. (Lyapunov) stable, if for all $\epsilon > 0$ there exists a scalar $\delta_1 \in (0, \epsilon)$ such that

$$\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1} < \delta_1 \implies \|y_k\|_{\mathcal{L}_\infty} < \epsilon, \quad \forall k \in \mathbb{N},$$

2. asymptotically stable, if it is Lyapunov stable and there exists $\delta_2 > 0$ such that $\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1} < \delta_2$ and $\mathbf{x}(0) \in c_0$ implies $\|y_k\|_{\mathcal{L}_\infty} \to 0$,

3. exponentially stable, if it is asymptotically stable, and there exists $\delta_3 > 0$ and continuous increasing functions $K : (0, 1) \to [1, \infty)$ and $\gamma : (0, 1) \to (0, 1)$ such that $\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda} < \delta_3$, implies

$$\|y_k\|_{\mathcal{L}_\infty} \le K(\lambda)\gamma(\lambda)^k (\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda}), \tag{6.4}$$

   for all $k \in \mathbb{N}$ and $\lambda \in (0, 1)$.

The motivation for the above definitions of stability is threefold:

1. The trajectories of the system will be affected by two different boundary conditions; the initial output vector y_0 and the initial state sequence $\mathbf{x}(0)$.

2. Since 0 is an equilibrium solution for (6.2), which is Lipschitz with respect to $(\chi(0), u)$, it is straightforward to show that the stability notions above translate directly to the state trajectory. For example, if the system is stable, given $\epsilon/(2L_1) > 0$ there exists a $\delta_1 > 0$ such that $\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1} < \delta_1$ implies $\|y_K\|_{\mathcal{L}_\infty} < \epsilon/(2L_1)$ for all $k \in \mathbb{N}$. Without loss of generality, we can choose these scalars so that $\delta_1 < \epsilon/(2L_1) < \delta$, where δ is the radius from Lemma 6.1 such that Lipschitz continuity holds. It follows that

$$\|x_{k+1}\|_{\mathcal{L}_\infty} \leq L_1(\|\mathbf{x}(0)\| + \|y_k\|_{\mathcal{L}_\infty}) < \epsilon, \quad \forall k \in \mathbb{N}.$$

3. For exponential stability, the dependency of the performance on the convergence speed λ of $\mathbf{x}(0)$ is expressed via the functions of K and γ, which are continuous and increasing to be physically meaningful. In addition, the right hand side of (6.4) scales linearly with the norm of the boundary $(y_0, \mathbf{x}(0))$, which is more in line with the corresponding definition for 1D systems, in contrast with [41, 140].

The definitions above also show the crucial difference between repetitive processes and 2D mixed continuous-discrete time systems [136, 137]; the latter covers the case where $T = \infty$ and studies the trajectory of the real vector $y_k(t)$ over $\mathbb{N} \times \mathbb{R}^+$, whereas we are concerned with the trajectory of the function y_k over $\mathbb{N}$. In linear repetitive process theory, the gap between these two topics are bridged via the notion of *stability along the pass* [2], which requires the stability parameters to be T independent, and translates to linear time invariant DRPs as the state space representation being Hurwitz. Although this property is desirable in experimental implementations or numerical simulations, as stated in Assumption 6.1 we forgo this requirement for theoretical purposes. Moreover, it is not clear how the time dependent functions f, g

can be extended to the positive real line for a given system where T is a priori known and fixed, making the issue of stability along the pass somewhat complicated. For a more detailed discussion of the relationship between repetitive processes and 2D systems theory, we invite the readers to consult [2].

We will say that the system is *globally* asymptotically (exponentially) stable if δ_2 (δ_3) can be chosen to be arbitrarily large. We will also be considering the case where $\mathbf{x}(0) = 0$. In the rest of the chapter, we will refer to any such DRP as a zero initial state (0-i.s.) system or process. The 0-i.s. system will be defined to be Lyapunov, asymptotically, or exponentially stable if the notions defined above hold for the case of $\mathbf{x}(0) = 0$; obviously the 0-i.s. system is (asymptotically/exponentially) stable if the actual system is (asymptotically/exponentially) stable. Also note that (6.4) is necessary *and* sufficient for 0-i.s exponential stability.

6.3 Lyapunov Theorems for DRPs

In this section we develop Lyapunov like theorems to assess the stability of 0-i.s. processes. The notion of a weak Lyapunov function [144], wherein the continuity assumption is weakened to continuity at the origin along with an annulus condition, which guarantees that the function is bounded away from 0, will be adapted to our case. Here we will omit the term weak and refer to such mappings as Lyapunov functionals. The exponential stability theorem at the end of the section will be used later to prove our main result. However, for completeness, we also present Lyapunov stability and asymptotic stability theorems.

A mapping $F : \mathcal{V} \to \mathbb{R}$, where $\mathcal{V}$ is a vector space equipped with the norm $\|.\|_V$, is said to satisfy the *annulus condition* if $F(0) = 0$, and there exists $\beta > 0$ such that $\sigma_F(\alpha) \triangleq \inf\{F(v) : v \in \mathcal{V}, \alpha \leq \|v\|_V \leq \beta\} > 0$ for all $\alpha \in (0, \beta)$.

Note that in finite dimensional spaces, continuity of V implies the annulus condition since the annulus is compact. As this is not the case for infinite dimensions, the annulus condition is rather necessary for our problem. In the following discussion, we will study the stability properties of the system (6.1) with 0-i.s., as a discrete nonlinear dynamical system of the form $y_{k+1} = \Gamma_0(y_k)$, where $\Gamma_0(.) \triangleq \pi_y(\Gamma(., 0))$, i.e. the input-output operator described by (6.2) with $\chi(0) = 0$. Consequently, a *Lyapunov functional* for (6.1) with 0-i.s. will be a mapping $V : \mathcal{Y} \to \mathbb{R}$ that 1) is continuous at 0, 2) satisfies the annulus condition, and 3) has a negative semidefinite difference $\Delta V(y) \triangleq V(\Gamma_0(y)) - V(y)$.

Theorem 6.1. *The nonlinear DRP* (6.1) *with 0-i.s. is stable if and only if it has a Lyapunov functional* $V : \mathcal{Y} \to \mathbb{R}$.

Proof. By Corollary 6.1, the mapping Γ_0 is continuous around 0. As a result, the proof extends from the finite dimensional case via theorems 1 and 2 of [144] without any modifications; since the finite dimensional aspect of the problem is not used in the proofs of these results. ■

Theorem 6.2. *The nonlinear DRP* (6.1) *with 0-i.s. is asymptotically stable if it has a Lyapunov functional* $V : \mathcal{Y} \to \mathbb{R}$ *so that* $(-\Delta V)$ *satisfies the annulus condition.*

Proof. Let V be a Lyapunov functional for (6.1) with 0-i.s. By Theorem 6.1, the system is stable. Now assume in addition that $-\Delta V$ satisfies the annulus condition. We will prove by contradiction that the origin is attractive, so assume the opposite. Select $\epsilon > 0$ such that the annulus condition for $-\Delta V$ holds in the ϵ ball. Let $\delta_1 > 0$ be small enough so that $\|y_0\|_{\mathcal{L}_\infty} < \delta_1$ implies $\|y_k\|_{\mathcal{L}_\infty} < \epsilon$. Then for every $\delta_2 \in (0, \delta_1)$ there is a y_0 with $\|y_0\|_{\mathcal{L}_\infty} < \delta_2$ so that the sequence $\{y_k\}_{k=0}^{\infty}$ does not converge to the origin. This means that there exists a $\bar{\epsilon} \in (0, \epsilon)$ such that for all N we have a $\kappa \geq N$ so $\bar{\epsilon} \leq \|x_\kappa\|_{\mathcal{L}_\infty} < \epsilon$. By the annulus condition, this implies that $V(y_\kappa) \geq \sigma_V(\bar{\epsilon}) > 0$. Furthermore, since ΔV satisfies the annulus condition it is negative definite, thus $V(y_k)$ is a decreasing sequence and by the monotone convergence theorem $V(y_k) \to V^* \geq \sigma_V(\bar{\epsilon})$. Note that this also means $V(y_k) \geq \sigma_V(\bar{\epsilon})$ for all $k \in \mathbb{N}$ since the sequence is nonincreasing. By the annulus condition, the supremum of ΔV on the annulus with radii $\bar{\epsilon}$ and ϵ is strictly negative. Thus for any κ that satisfies $\|y_\kappa\|_{\mathcal{L}_\infty} \geq \bar{\epsilon}$, $V(y_{\kappa+1}) \leq V(y_\kappa) - \sigma_{-\Delta V}(\bar{\epsilon})$. But this means that V must eventually become smaller than $\sigma(\bar{\epsilon})$, contradicting our assumption that for all N there exists a $\kappa \geq N$ so $\|y_\kappa\|_{\mathcal{L}_\infty} \geq \bar{\epsilon}$. ■

Theorem 6.3. *The nonlinear DRP* (6.1) *with 0-i.s. is exponentially stable if and only if there exists $V : \mathcal{Y} \to \mathbb{R}$ and positive scalars c_1, c_2, c_3, with $c_2 > c_3$ such that V satisfies*

1. $c_1\|y\|_{\mathcal{L}_\infty} \leq V(y) \leq c_2\|y\|_{\mathcal{L}_\infty}$, *and*

2. $\Delta V(y) \leq -c_3\|y\|_{\mathcal{L}_\infty}$,

in a neighborhood of the origin.

Proof. Sufficiency is rather obvious through some algebraic manipulations and is therefore omitted. For necessity, assume that the system is exponentially stable, then there exists $K > 1$, $\delta_3 > 0$ and $\gamma \in [0, 1)$ so that $\|\Gamma_0^k(y)\|_{\mathcal{L}_\infty} \leq K\gamma^k\|y\|_{\mathcal{L}_\infty}$ holds for all $y \in \mathcal{Y}$ with $\|y_0\| < \delta_3$. Let N be an integer such that $K\gamma^N < 1$. Define

$$V(y) \triangleq \sum_{i=0}^{N-1} \|\Gamma_0^i(y)\|_{\mathcal{L}_\infty} \geq \|y\|_{\mathcal{L}_\infty}$$

for all $y \in \mathcal{Y}$ with $\|y\|_{\mathcal{L}_\infty} < \delta_3$, which implies

$$\Delta V(y) = V(\Gamma_0(y)) - V(y) = \|\Gamma_0^N(y)\|_{\mathcal{L}_\infty} - \|y\|_{\mathcal{L}_\infty},$$

for all $y \in \mathcal{Y}$ with $\|y\|_{\mathcal{L}_\infty} < \delta_3$. By exponential stability, it follows that

$$V(y) \leq K\frac{1-\gamma^N}{1-\gamma}\|y\|_{\mathcal{L}_\infty},$$
$$\Delta V(y) \leq -(1-K\gamma^N)\|y\|_{\mathcal{L}_\infty},$$

for all $y \in \mathcal{Y}$ with $\|y\|_{\mathcal{L}_\infty} < \delta_3$. Finally, we note that

$$(1-K\gamma^N) < 1 < K\frac{1-\gamma^N}{1-\gamma},$$

to conclude the proof. ■

6.4 Stability of LTV Differential Processes

In this section we will focus on systems where f and g are linear with respect to their first two arguments for fixed $t \in [0,T]$, and relax the continuous differentiability assumption to that of continuity; i.e. we will look at LTV differential processes of the form

$$\begin{aligned} \dot{x}_{k+1}(t) &= A(t)x_{k+1}(t) + B(t)y_k(t), \\ y_{k+1}(t) &= C(t)x_{k+1}(t) + D(t)y_k(t), \end{aligned} \tag{6.5}$$

for all $t \in [0,T]$ and $k \in \mathbb{N}$, where A, B, C, D are continuous real matrices of appropriate size.

6.4.1 0-i.s. Stability and the Spectral Radius

Similar to the nonlinear case, given the LTV system (A,B,C,D), we denote by G_x the state response to the input and the initial condition, and by G_y the mapping from the input and the state to the output. The LTV operator G is defined so that

$$(w,\chi) = G(u,\chi(0)) \triangleq G_y(u, G_x(u,\chi(0))), G_x(u,\chi(0)),$$

and the 0-i.s. output response $G_0(.) \triangleq \pi_y(G(.,0))$. As before, we first consider the 0-i.s. system, which can be described by the discrete dynamical system $y_{k+1} = G_0 y_k$ on $\mathcal{Y}$. We have the following claim about LTV operators like G_0:

Claim 6.3. *Any LTV input-output operator represented by continuous state matrices over a finite time horizon is bounded with respect to the $\mathcal{L}_p$ norm, for all $p \in [1, \infty]$.*

Claim 6.3 makes some intuitive sense since linear systems do not have finite escape time; the formal proof of this argument is given in Appendix B.2. As such, we will expand the space $\mathcal{Y}$ to $\mathcal{L}_\infty$, and more generally $\mathcal{L}_p$. Moreover, as G_0 is bounded and linear, it is continuous, and the results of previous section can be used to assess the stability of the 0-i.s. system. However, as the readers can guess, the stability problem is much simpler for the linear system, and exponential stability can be conveniently expressed as a spectral radius condition:

Theorem 6.4. *The 0-i.s. system is exponentially stable (in $\mathcal{L}_p$) if and only if the spectrum of G_0 is contained in the interior of the open unit circle.*

Proof. This follows from Gelfand's spectral radius formula $\rho(G_0) = \lim_{k\to\infty} \|G_0^k\|_{\mathcal{L}_p}^{1/k}$ and is omitted for brevity. ■

Remark 6.1. In general, the condition $\rho(G_0) < 1$ is sufficient for asymptotic stability, whereas $\rho(G_0) \leq 1$ is necessary [145]. This issue is circumvented in [2] (page 44), by requiring asymptotic stability to be a local property around a nominal operator.

6.4.2 Computation of the Spectral Radius

The computation of the spectral radius will be similar to the procedure outlined for the time invariant case in [2]. Let $P_z(t) \triangleq zI - D(t)$, where $z \in \mathbb{C}$ and consider the operator $(zI - G_0)$ which maps u to η, given by

$$\dot{\chi}(t) = A(t)\chi(t) + B(t)u(t),$$
$$w(t) = C(t)\chi(t) + D(t)u(t),$$
$$\eta(t) = zu(t) - w(t),$$

which yields

$$\begin{aligned} \dot{\chi}(t) &= A(t)\chi(t) + B(t)u(t), \\ \eta(t) &= -C(t)\chi(t) + P_z(t)u(t). \end{aligned} \tag{6.6}$$

for all $t \in [0, T]$. If $|z| > \sup_{t\in[0,\tau)} \rho(D(t))$, then

$$\dot{\chi}(t) = (A(t) + B(t)P_z^{-1}(t)C(t))\chi(t) + P_z^{-1}(t)\eta(t),$$
$$u(t) = P_z^{-1}(t)(C(t)\chi(t) + \eta(t)),$$

for all $t \in [0, \tau)$, where $\tau \in (0, T)$. Thus, $(zI - G_0)$ is invertible if

$$|z| > \alpha \triangleq \max_{t\in[0,T]} \rho(D(t)).$$

In addition, this also implies $(zI - G_0)^{-1}$ is bounded (in $\mathcal{L}_p$) by the bounded inverse theorem. Hence, the spectrum is contained within a closed disk of radius α.

Otherwise, given any $\epsilon > 0$, let $z \in \mathbb{C}$ be a number such that $P_z(t)$ is singular for some $t \in [0, T)$ and $|z| > \alpha - \epsilon$. Such a z exists since the spectral radius of D varies continuously. Define $s \triangleq \min\{t \in [0, T] : \det(P_z(t)) = 0\}$, and set $\eta(t) = \varphi\mathbf{1}(t - s)$, where φ is orthogonal to the range of $P_z(s)$, and $\mathbf{1}(.)$ is the Heaviside step function. Assume that there exists a $u \in \mathcal{L}_\infty$ that achieves η almost everywhere. Obviously, the input $u = 0$ and state $\chi = 0$, almost everywhere on $[0, s)$. Define

$$\mu(t) \triangleq \|\varphi - P_z(t)u(t) + C(t)\chi(t)\|_2, \quad \forall t \in [s, T].$$

By (6.6), $\mu = 0$ almost everywhere on $[s, T]$. Moreover, since χ is continuous[25] by (6.6), $\chi(s) = 0$. Now let Ψ be an orthogonal projection matrix, onto the span of φ. Using the reverse triangle inequality, by orthogonality we have

$$\mu(t) \geq \sqrt{\|\varphi - \Psi P_z(t)u(t)\|_2^2 + \|(I - \Psi)P_z(t)u(t)\|_2^2} - \|C(t)\chi(t)\|_2,$$

and thus

$$\mu(t) \geq \|\varphi - \Psi P_z(t)u(t)\|_2 - \|C(t)\chi(t)\|_2 \geq \|\varphi\|_2 - \left(\|\Psi P_z(t)u(t)\|_2 + \|C(t)\chi(t)\|_2\right),$$

for all $t \in [0, T]$. Clearly, $\mu(s) \geq \|\varphi\|_2$. In addition, since P_z, C, χ are continuous, $\chi(s) = 0$, and $\Psi P_z(s) = 0$,

$$\sup_{\tau \in [s,t)} \left(\|\Psi P_z(\tau)\|_2 + \|C(\tau)\chi(\tau)\|_2\right),$$

can be made arbitrarily small as t approaches s from the right. Consequently, given any $u \in \mathcal{L}_\infty$, the essential supremum of $\|C(\tau)\chi(\tau)\|_2 + \|\Psi P_z(\tau)u(\tau)\|_2$ can be made arbitrarily small almost everywhere on $[s, t)$ as t approaches s from the right. But then, $\mu(t) \geq \varsigma > 0$, almost everywhere on $[s, t)$ for some $t > s$ and constant ς, contradicting the fact that $\mu = 0$. It follows that $(zI - G_0)$ is not surjective and thus $(zI - G_0)^{-1}$ does not exist. In other words, there exist spectral values of G_0 within the open ball of radius α, arbitrarily close to the closed disk of radius α. Therefore, $\rho(G_0) = \max_{t \in [0,T]} \rho(D(t))$.

6.4.3 Stability under Nonzero Initial States

Let $H \triangleq \pi_x \circ G$ be the natural response of the LTV system to initial conditions, where π_x is the canonical projection onto the space of the state trajectory. Then the

[25] See [146], page 48, for the case of piecewise continuous u; and [147], theorem II.4.6 for the case of integrable u.

solution of (6.5) can be given as

$$y_n = G_0^n y_0 + \sum_{i=1}^{n} G_0^{n-i} H x_i(0), \quad n \in \mathbb{N}.$$

Now if $\rho(G_0) > 1$, by Gelfand's spectral radius formula, there exist scalars $M > 0$ and $\zeta \in (0,1)$ such that $\|G_0^k\|_{\mathcal{L}_\infty} \leq M\zeta^k$ for all $k \in \mathbb{N}$. Therefore,

$$\|y_k\|_{\mathcal{L}_\infty} \leq M\zeta^k \|y_0\|_{\mathcal{L}_\infty} + M\|H\|_{\mathcal{L}_\infty} \sum_{i=1}^{k} \zeta^{k-i} \|x_i(0)\|, \tag{6.7}$$

for all $k \in \mathbb{N}$, where H is bounded due to the finite time assumption[26]. If $\|\mathbf{x}(0)\|_{e_1}$ is finite, we have

$$\begin{aligned} \|y_k\|_{\mathcal{L}_\infty} &\leq M\left(\zeta^k \|y_0\|_{\mathcal{L}_\infty} + \|H\| \|\mathbf{x}(0)\|_{e_1} \textstyle\sum_{i=1}^{k} \zeta^{k-i}\right) \\ &= M\left(\zeta^k \|y_0\|_{\mathcal{L}_\infty} + \|H\|_{\mathcal{L}_\infty} \|\mathbf{x}(0)\|_{e_1} \tfrac{1-\zeta^k}{1-\zeta}\right) \\ &\leq M \max\left\{1, \tfrac{\|H\|_{\mathcal{L}_\infty}}{1-\zeta}\right\} \left(\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1}\right), \end{aligned}$$

for all $k \in \mathbb{N}$. Thus, the LTV system is stable. Now assume in addition $\mathbf{x}(0) \in c_0$, and consider the partial sum in the second term of the right hand side of (6.7),

$$S_k \triangleq \sum_{i=1}^{k} \zeta^{k-i} \|x_i(0)\| \geq 0, \quad \forall k \in \mathbb{N}.$$

Then, it is easy to verify $S_{k+1} = \zeta S_k + \|x_{k+1}(0)\| \geq 0$ for all $k \in \mathbb{N}$, so $S_k \to 0$ by Claim 6.1. Therefore, we can conclude by (6.7) that $y_k \to 0$ if $\mathbf{x}(0) \in c_0$ and $\rho(G) < 1$.

Finally, we consider the case where $\mathbf{x}(0) \in e_\lambda$. From (6.7)

$$\begin{aligned} \|y_k\| &\leq M\zeta^k \|y_0\|_{\mathcal{L}_\infty} + M\|H\|_{\mathcal{L}_\infty} \|\mathbf{x}(0)\|_{e_\lambda} \sum_{i=1}^{k} \zeta^{k-i} \lambda^{i-1} \\ &\leq M\left(\zeta^k \|y_0\|_{\mathcal{L}_\infty} + \|H\|_{\mathcal{L}_\infty} \|\mathbf{x}(0)\|_{e_\lambda} k \bar{\lambda}^{k-1}\right) \end{aligned}$$

[26]See the proof of Claim 6.3 to verify that the relevant state transition matrix is bounded.

where $\bar{\lambda} \triangleq \max\{\zeta, \lambda\}$, so by Claim 6.2

$$\|y_k\| \leq M\zeta^k \|y_0\|_{\mathcal{L}_\infty} + M\|H\|_{\mathcal{L}_\infty} \|\mathbf{x}(0)\|_{e_\lambda} \frac{2}{1-\bar{\lambda}} \left(\frac{1+\bar{\lambda}}{2}\right)^k,$$

and since $\zeta \leq \bar{\lambda} < (1+\bar{\lambda})/2 < 1$,

$$\|y_k\|_{\mathcal{L}_\infty} \leq \overbrace{M \max\left\{1, \frac{2\|H\|_{\mathcal{L}_\infty}}{1-\bar{\lambda}}\right\}}^{K_G(\bar{\lambda})} \times \left(\underbrace{(1+\bar{\lambda})/2}_{\gamma_G(\bar{\lambda})}\right)^k (\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda}), \quad \forall k \in \mathbb{N}. \quad (6.8)$$

Noting that $K_G(\max\{\zeta, \lambda\})$ and $\gamma_G(\max\{\zeta, \lambda\})$ defined in (6.8) are both continuous and increasing in λ, we can conclude the system to be exponentially stable.

The findings of the section can be summarized as follows:

Theorem 6.5. *For the LTV DRP* (6.5)*, the following are equivalent:*

1. $\rho(G_0) = \max_{t\in[0,T]} \rho(D(t)) < 1$.
2. *The 0-i.s. system is globally exponentially stable.*
3. *The system is globally exponentially stable.*

Remark 6.2. Note that the analysis of Section 6.4.3 extends directly to any $\mathcal{L}_p$ norm topology since the spectrum of G_0 is contained in a closed disk of radius

$$\max_{t\in[0,T]} \rho(D(t))$$

regardless of the choice of $\mathcal{L}_p$ norm. Therefore, $\max_{t\in[0,T]} \rho(D(t)) < 1$ implies global exponential stability in $\mathcal{L}_p$.

6.5 Linearized Stability of DRPs

In this section, we will establish the equivalence between exponential stability of a nonlinear DRP of the form (6.1) with that of its linearization. The linearization of (6.1) will mirror that of the 1D case, in other words, we will be linearizing the differential operator (6.2) as is typical in feedback control. This will be done as follows: Since f and g are continuously differentiable,

$$\dot{\chi}(t) = \underbrace{\bar{A}(t)\chi(t) + \bar{B}(t)u(t) + b(\chi(t), u(t), t)}_{f(\chi(t),u(t),t)}, \tag{6.9}$$

and

$$w(t) = \underbrace{\bar{C}(t)\chi(t) + \bar{D}(t)u(t) + d(\chi(t), u(t), t)}_{g(\chi(t),u(t),t)}, \tag{6.10}$$

for some continuous functions b and d, as

$$\begin{aligned} \bar{A}(t) &\triangleq \frac{\partial f}{\partial \chi}(0,0,t), \qquad & \bar{B}(t) &\triangleq \frac{\partial f}{\partial u}(0,0,t), \\ \bar{C}(t) &\triangleq \frac{\partial g}{\partial \chi}(0,0,t), \qquad & \bar{D}(t) &\triangleq \frac{\partial g}{\partial u}(0,0,t), \end{aligned}$$

are continuous. Consequently, the linearization of (6.1) will be defined as the following 2D system:

$$\begin{aligned} \dot{\bar{x}}_{k+1}(t) &= \bar{A}(t)\bar{x}_{k+1}(t) + \bar{B}(t)\bar{y}_k(t), \\ \bar{y}_{k+1}(t) &= \bar{C}(t)\bar{x}_{k+1}(t) + \bar{D}(t)\bar{y}_k(t), \end{aligned} \tag{6.11}$$

for all $t \in [0, T]$ and $k \in \mathbb{N}$, with boundary conditions $\bar{\mathbf{x}}(0) = \mathbf{x}(0)$ and $\bar{y}_0 = y_0$.

6.5.1 Asymptotics of the Nonlinear Perturbations

Let f_i be the i-th output of f. Since f is continuously differentiable in Z and f vanishes at the origin uniformly in time, i.e. $f(0, 0, t) = 0$, by the multivariable mean value theorem, there exists a point (ξ_i^*, υ_i^*) on the line segment connecting (ξ, υ) to the origin such that

$$f_i(\xi, \upsilon, t) = \begin{bmatrix} \frac{\partial f_i}{\partial \xi}(\xi_i^*, \upsilon_i^*, t) & \frac{\partial f_i}{\partial \upsilon}(\xi_i^*, \upsilon_i^*, t) \end{bmatrix} \begin{bmatrix} \xi \\ \upsilon \end{bmatrix},$$

in a neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}^m$. Equivalently,

$$\begin{aligned} f_i(\xi, \upsilon, t) = & \begin{bmatrix} \bar{A}_i(t) & \bar{B}_i(t) \end{bmatrix} \begin{bmatrix} \xi \\ \upsilon \end{bmatrix} \\ & + \underbrace{\begin{bmatrix} \left(\frac{\partial f_i}{\partial \xi}(\xi_i^*, \upsilon_i^*, t) - \bar{A}_i(t) \right) & \left(\frac{\partial f_i}{\partial \upsilon}(\xi_i^*, \upsilon_i^*, t) - \bar{B}_i(t) \right) \end{bmatrix} \begin{bmatrix} \xi \\ \upsilon \end{bmatrix}}_{b_i(\xi, \upsilon, t)}, \end{aligned}$$

where $\bar{A}_i$ and $\bar{B}_i$ are the i-th rows of $\bar{A}$ and $\bar{B}$, respectively, and b_i is the i-th output of b. Now let $q_i \triangleq \partial f_i / \partial \xi$. The function q_i is continuous in Z because f is continuously differentiable in Z. Hence, by the Heine-Cantor theorem, q_i is uniformly continuous in Z. Therefore, for all $\epsilon > 0$ there exists $\delta_o > 0$ such that

$$\|(\xi, \upsilon)\| < \delta_o, \implies \|q_i(\xi, \upsilon, t) - \bar{A}_i(0, 0, t)\| < \epsilon,$$

for every $(\xi, \upsilon, t) \in Z$, since $q_i(0, 0, t) = \bar{A}_i(0, 0, t)$. Using similar arguments for the other partial derivatives $\partial f_i / \partial \upsilon, \partial g_i / \partial \xi, \partial g_i / \partial \upsilon$, we can conclude that for all $\epsilon > 0$ there exists $\delta_O > 0$ satisfying

$$\|(\xi, \upsilon)\| < \delta_O, \implies \|(b(\xi, \upsilon, t), d(\xi, \upsilon, t)\| < \epsilon \|(\xi, \upsilon)\|, \quad \forall (\xi, \upsilon, t) \in Z. \qquad (6.12)$$

6.5.2 $\mathcal{L}_\infty$ Asymptotics of the Linearization Error

Next, let us consider the LTV system defined by the matrices $\bar{A}, \bar{B}, \bar{C}, \bar{D}$:

$$\begin{aligned} \dot{\bar{\chi}}(t) &= \bar{A}(t)\bar{\chi}(t) + \bar{B}(t)\bar{u}(t), \\ \bar{w}(t) &= \bar{C}(t)\bar{\chi}(t) + \bar{D}(t)\bar{u}(t), \end{aligned} \tag{6.13}$$

for all $t \in [0, T]$, where $\bar{\chi}(0) = \chi(0)$. The 0-i.s. input-output operator $\bar{G}_0$ and the initial state response operator $\bar{H}$ will be defined for this system as in Section 6.4. Subtracting (6.13) from (6.9) and (6.10)

$$\begin{aligned} \dot{\tilde{\chi}}(t) &= \bar{A}(t)\tilde{\chi}(t) + \bar{B}(t)\tilde{u}(t) + b(\chi(t), u(t), t), \\ \tilde{w}(t) &= \bar{C}(t)\tilde{\chi}(t) + \bar{D}(t)\tilde{u}(t) + d(\chi(t), u(t), t), \end{aligned}$$

where $\tilde{\chi}(t) \triangleq \chi(t) - \bar{\chi}(t)$, $\tilde{w}(t) \triangleq w(t) - \bar{w}(t)$, and similarly $\tilde{u}(t) \triangleq u(t) - \bar{u}(t)$. Define the mapping φ so that

$$(\varphi(\chi, u))(t) = (b(\chi(t), u(t), t), d(\chi(t), u(t), t)).$$

Then the output error $\tilde{w}$ is given by

$$\tilde{w} = \bar{G}_0\tilde{u} + \Omega(\varphi(\chi, u)), \tag{6.14}$$

where Ω represents the $\mathcal{L}_\infty$ stable input-output response of an LTV system with the state matrices

$$\left(A, \begin{bmatrix} I & 0 \end{bmatrix}, C, \begin{bmatrix} 0 & I \end{bmatrix}\right).$$

The following lemma will define the asymptotic behavior of φ with respect to $(u, \chi(0))$.

Lemma 6.2. *For all $\epsilon > 0$, there exists $\delta^* > 0$ such that*

$$\|u\|_{\mathcal{L}_\infty} + \|\chi(0)\| < \delta^*, \implies \|\varphi(\chi, u))\|_{\mathcal{L}_\infty} \leq \epsilon(\|u\|_{\mathcal{L}_\infty} + \|\chi(0)\|).$$

6.5.3 Necessary and Sufficient Conditions for Exponential Stability

We first assume that the 0-i.s. linear system is exponentially stable so $\|\bar{G}_0^k\|_{\mathcal{L}_\infty} \leq \bar{M}\bar{\zeta}^k$ for all $k \in \mathbb{N}$, for some $\bar{M} \geq 1, \bar{\zeta} \in (0, 1)$. With this, let $N \in \mathbb{N}$ such that $\bar{M}\bar{\zeta}^N < 1$. We will need the following result:

Lemma 6.3. *There exist scalars $\delta_{\text{fh}} > 0$ and $L_{\text{fh}} \geq 1$ so that $\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1} < \delta_{\text{fh}}$ implies*

$$\|y_k\|_{\mathcal{L}_\infty} < L_{\text{fh}}(\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1})$$

for all $k \in \{0, 1, \ldots, N-1\}$.

Proof. The proof follows easily from Lipschitz continuity of $\pi_y \circ \Gamma$ (Corollary 6.1) and is omitted for brevity. ■

Proposition 6.1. *The nonlinear system is exponentially stable if its linearization is exponentially stable.*

The proof of this proposition is rather involved and as such given in Appendix B.3 for a more compact presentation. To establish the converse of this result, we will follow an indirect route that is much easier compared to a direct proof. Specifically, we will show that nonlinear exponential stability implies linear exponential stability for the 0-i.s case. This will allow us to finalize our main result by aid of Theorem 6.5, as the reader can see in Figure 6.2.

Proposition 6.2. *The linearization of the nonlinear system is 0-i.s. exponentially stable if the nonlinear system is 0-i.s. exponentially stable.*

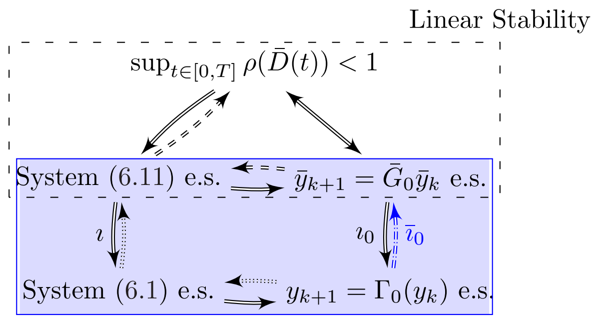


Figure 6.2: Implication diagram for exponential stability (e.s.): The linear exponential stability diagram was stated in Theorem 6.5, where the dashed implication arrows were established by proving the solid ones. For the nonlinear case, implications $\imath, \imath_0$ are proven in Proposition 6.1. Proving implication $\bar{\imath}_0$ will close the loop and allow us to conclude the dotted implication arrows.

Proof. Let V be the Lyapunov function from Theorem 6.3. Then

$$c_1\|y\|_{\mathcal{L}_\infty} \leq V(y) \leq c_2\|y\|_{\mathcal{L}_\infty},$$

and the difference of V with respect to the linear operator $\bar{G}_0$ is

$$\begin{aligned}\Delta V(y) &= V(\bar{G}_0 y) - V(y)\\ &= (V(\bar{G}_0 y) - V(\Gamma_0(y))) + (V(\Gamma_0(y)) - V(y))\\ &\leq (V(\bar{G}_0 y) - V(\Gamma(y))) - c_3\|y\|_{\mathcal{L}_\infty},\end{aligned}$$

for some positive c_1, c_2, c_3, with $c_2 > c_3$, since the nonlinear DRP is exponentially stable, around the origin. The function V is Lipshitz because it is a sum of Lipschitz functions; Γ_0^i is Lipschitz for any $i \in \mathbb{N}$. Therefore, there exists a positive scalar L_G such that

$$|V(\Gamma_0(y)) - V(\bar{G}_0 y)| \leq L_G\|\bar{G}_0 y - \Gamma_0(y)\|_{\mathcal{L}_\infty},$$

around the origin. Furthermore, from (6.14),

$$\bar{G}_0 y - \Gamma_0(y) = \Omega(\varphi(\Gamma_x(y, 0), y)).$$

Hence, for any $\epsilon > 0$, by Lemma 6.2, there exists $\delta^* > 0$ so that $\|y\|_{\mathcal{L}_\infty} < \delta^*$ implies

$$|V(\Gamma_0(y)) - V(\bar{G}_0 y)| \leq \epsilon \|y\|_{\mathcal{L}_\infty},$$

and therefore for any $\bar{c}_3 \in (0, c_3)$, there exists a $\bar{\delta}_3 > 0$ so that $\|y\|_{\mathcal{L}_\infty} \leq \bar{\delta}_3$ implies

$$\Delta V(y) \leq (V(\bar{G}_0 y) - V(\Gamma(y))) - c_3 \|y\|_{\mathcal{L}_\infty} \leq \bar{c}_3 \|y\|_{\mathcal{L}_\infty}.$$

■

With this, we can state our main result, which summarizes the findings of Theorem 6.5 and Propositions 6.1 and 6.2 as given below:

Theorem 6.6. *For the nonlinear DRP* (6.1) *and its linearization* (6.11)*, the following are equivalent:*

1. *The condition* $\max_{t\in[0,T]} \rho(\bar{D}(t)) < 1$ *holds.*
2. *The 0-i.s. DRP* (6.11) *is globally exponentially stable.*
3. *The DRP* (6.11) *is globally exponentially stable.*
4. *The 0-i.s. DRP* (6.1) *is exponentially stable.*
5. *The DRP* (6.1) *is exponentially stable.*

6.6 Applications: Picard Iterations and ILC

We now present two applications of Theorem 6.6.

6.6.1 Picard Iterates with Varying Initial Conditions

The Picard-Lindelöf theorem guarantees the existence and uniqueness of the solution x^* of the differential equation $\dot{x}(t) = f(x(t), t)$ with initial condition $x^*(0) = x_0^*$

for small T. The existence of this solution is proven by a recursive process, whose convergence is shown by the contraction mapping theorem. These iterates can be expressed as the DRP

$$\begin{aligned} \dot{x}_{k+1}(t) &= f(y_k(t), t), \quad x_{k+1}(0) = x_0^* \\ y_{k+1}(t) &= x_{k+1}(t), \end{aligned}$$

for all $t \in [0, T]$ and $k \in \mathbb{N}$. The time varying transformation

$$(x_k(t), y_k(t)) \mapsto (x_k(t) - x^*(t), y_k(t) - x^*(t))$$

translates the equilibrium to 0, uniformly in time:

$$\begin{aligned} \dot{\underline{x}}_{k+1}(t) &= \underline{f}(\underline{y}_k(t), t), \quad \underline{x}_{k+1}(0) = 0 \\ \underline{y}_{k+1}(t) &= \underline{x}_{k+1}(t), \end{aligned}$$

with $\underline{f}(\chi, t) \triangleq f(\chi + x^*(t), t) - \dot{x}^*(t)$, for all $t \in [0, T]$ and $k \in \mathbb{N}$. This resulting system satisfies continuous differentiability assumptions around the new equilibrium since the fixed point x^* is *twice* continuously differentiable by virtue of f being continuously differentiable. Now, we can conclude that Picard iterates form an exponentially stable DRP when $y_0 - x^*$ and $\mathbf{x}(0) - x_0^*$ are small enough. One implication of this result is that the iterates converge to x^* for every $\mathbf{x}(0)$ with $\mathbf{x}(0) - x_0^* \in c_0$, e.g. for *nonconstant initial state sequences that converge to* x_0, when the boundary conditions are sufficiently close to the equilibrium.

6.6.2 ILC with Static Nonlinear Update Laws

The second application of Theorem 6.6 addresses the ILC problem of iteratively constructing the input u^* given a desired output y_{des} so

$$\dot{x}^*(t) = f(x^*(t), u^*(t), t),$$
$$y_{\text{des}}(t) = g(x^*(t), u^*(t), t),$$

for all $t \in [0, T]$. We consider the ILC system, where l satisfies $l(0, t) = 0$,

$$\dot{x}_{k+1}(t) = f(x_{k+1}(t), u_{k+1}(t), t),$$
$$y_{k+1}(t) = g(x_{k+1}(t), u_{k+1}(t), t),$$
$$u_{k+1}(t) = u_k(t) + l(e_k(t), t),$$

and $e_k \triangleq y_k - y_{\text{des}}$, for all $t \in [0, T]$ and $k \in \mathbb{N}$. This static (in time) update law is based on the internal model principle in the iteration domain, and guarantees perfect tracking in the limit for all achievable y_{des} when stable. Following a transformation akin to the one for Picard iterates, details of which are skipped, we can rewrite the system as

$$\dot{\underline{x}}_{k+1}(t) = \underline{f}(\underline{x}_{k+1}(t), \underline{u}_k(t), e_k(t), t),$$
$$\begin{bmatrix} e_{k+1}(t) \\ \underline{u}_{k+1}(t) \end{bmatrix} = \begin{bmatrix} \underline{g}(\underline{x}_{k+1}(t), \underline{u}_k(t), e_k(t), t), \\ \underline{u}_k(t) + l(e_k(t), t) \end{bmatrix},$$

with

$$\underline{g}(\chi, u, \theta, t) \triangleq g(\chi + x^*(t), u + u^*(t) + l(\theta, t), t) - y_{\text{des}}(t),$$

for all $t \in [0, T]$ and $k \in \mathbb{N}$. Note that e_0 depends on $\underline{u}_0$, so $(e_0, \underline{u}_0)$ cannot be arbitrarily chosen, and therefore it is difficult to derive necessary exponential stability conditions. Nevertheless, by taking the appropriate partials and letting

$$\underline{D}(t) \triangleq \frac{\partial g}{\partial u}(x^*(t), u^*(t), t), \quad \underline{L}(t) \triangleq \frac{\partial l}{\partial \theta}(0, t),$$

for all $t \in [0, T]$, the system is exponentially stable if

$$\max_{t\in[0,T]} \rho \left(\begin{bmatrix} \underline{D}(t) \\ I \end{bmatrix} \begin{bmatrix} \underline{L}(t) & I \end{bmatrix} \right) = \max_{t\in[0,T]} \rho(I + \underline{L}(t)\underline{D}(t)) < 1,$$

where the equality can be verified via simple eigenvector manipulations, with the equivalent condition being $\max_{t\in[0,T]} \rho(I + \underline{D}(t)\underline{L}(t)) < 1$ for square systems. The significance of this result stems from the following:

1. The stability condition is consistent with the monotonic convergence condition for single input single output systems given in [55]. For multiple-input multiple-output systems, it is much simpler than the integral condition in [55] and easily computable.

2. It shows that the error term in the learning function l must be replaced with its $\bar{n}$-th derivative for a vector relative degree $\bar{n}$ system, and the desired output should be sufficiently smooth, in line with [148].

3. It shows that initial condition errors affect the tracking error continuously, and perfect tracking can be achieved asymptotically (exponentially) when the initial condition errors vanish asymptotically (exponentially), as per [40, 103].

4. As opposed to the conventional norm based conditions, which rely on the time weighted norm,[27] this is the first spectral radius based stability condition in the nonlinear ILC literature, which is naturally less conservative. In addition,

[27] This norm is equivalent to the $\mathcal{L}_\infty$ norm in finite time and is the $\mathcal{L}_\infty$ norm of the exponential weighting of a function.

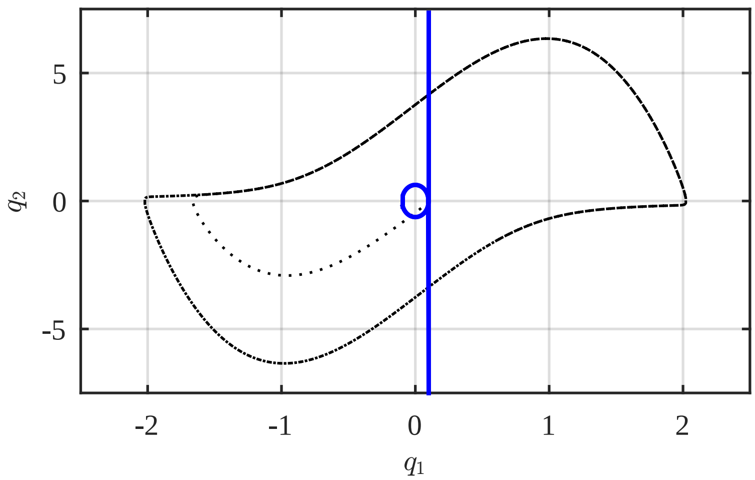


Figure 6.3: Limit cycle of the unforced oscillator with the damping $\Xi(t) = 4$ (dotted black line), and the learned state trajectory of the time varying actuated oscillator after 30 trials (solid blue line).

to the best of our knowledge, it is among the first studies of ILC from a local perspective, so that nonlinear time varying update laws can be considered for locally continuously differentiable systems in lieu of adaptive laws, without resorting to saturation [143].

5. In [9], the authors note "ILC designs using discrete-time linearizations of nonlinear systems often yield good results when applied to the nonlinear systems". The condition we have found verifies that linearization is indeed a valid strategy with continuous proportional or derivative type ILC algorithms.

To close this section, we note that the same methodology can be used to derive spectral stability conditions when the input is Q filtered; i.e. the update law is of the form $u_{k+1}(t) = \underline{Q}(t)u_k(t) + l(e_k(t), t)$, which is known to be a stabilizing factor against numerical errors in ILC algorithms.

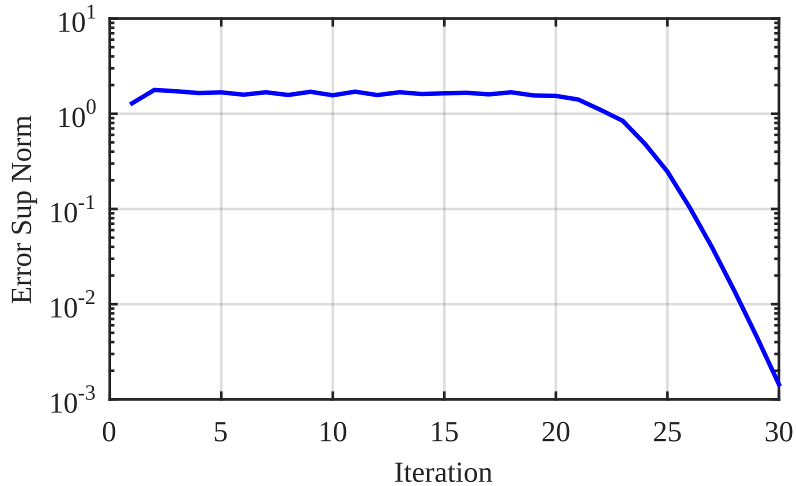


Figure 6.4: Evolution of $\|e_k\|_{\mathcal{L}_\infty}$.

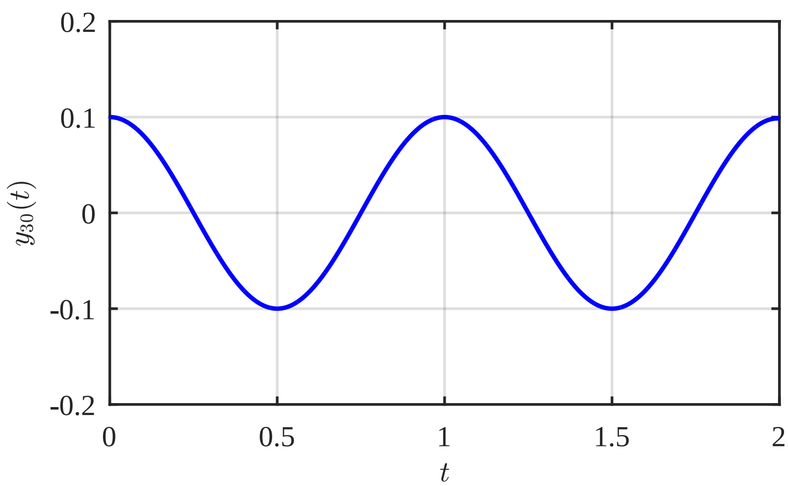


Figure 6.5: The learned output signal after 30 trials.

6.7 Illustrative Example

Consider the actuated Van der Pol oscillator in normal form with a time varying damping coefficient:

$$
\begin{aligned}
\dot{q}_1(t) &= q_2(t), \\
\dot{q}_2(t) &= -q_1(t) + \Xi(t)(1 - (q_1(t))^2)q_2(t) + u(t), \\
y &= q_1(t),
\end{aligned}
$$

where the damping coefficient $\Xi(t) > 0$, and $t \in [0, 2]$. The unforced oscillator is well known to have an unstable equilibrium at the origin for all constant $\Xi(t) > 0$; the trajectory of the unforced system with $\Xi(t) = 4$ is plotted in Figure 6.3. Our objective is to track the desired reference $y_{\mathrm{des}}(t) = 0.1\cos(2\pi t)$. As the oscillator has relative degree 2, we consider the update law

$$u_{k+1}(t) = u_k(t) - (\ddot{y}_k(t) - \ddot{y}_{\mathrm{des}}(t)), \quad \forall t \in [0, T], k \in \mathbb{N}.$$

Then it is easy to verify that this update law is stable since

$$\ddot{y}(t) = \dot{q}_2(t), \partial\dot{q}_2(t)/\partial u(t) = 1.$$

Indeed, for $\Xi(t) = 4 + 0.5\sin(2\pi(10t))$, Figure 6.4 shows that the tracking error is exponentially decreased when $u_0 = 0$ and the initial conditions are randomly chosen to exponentially converge to $(y_{\mathrm{des}}(0), \dot{y}_{\mathrm{des}}(0)) = (0.1, 0)$ with convergence rate λ (also randomly chosen) and e_λ norm less than 0.1. The learned state trajectory after 30 trials is shown in contrast to the limit cycle of the system in Figure 6.3, the learned output signal tracks the desired output with very high precision as can be seen in Figure 6.5, *without any stabilizing feedback.*

6.8 Conclusion

In this chapter, we addressed the problem of finding necessary and sufficient exponential stability conditions for a class of nonlinear repetitive processes. Assuming existence, uniqueness and boundedness of the integral curves of the associated vector fields for small signals and initial conditions, we showed Lipschitz continuity of the nonlinear operator in finite time. Based on this, we developed Lyapunov stability theorems for DRPs treated as nonlinear recursion in the input space. In the sequel, we approximated the system as an LTV system, and showed that the approximation error is $o(y_k)$ in the $\mathcal{L}_\infty$ norm as y_k tends to 0. Using this asymptotic property, along with the Lyapunov theorems, we showed that a DRP is exponentially stable if and only if the state matrix $\bar{D}$ is uniformly Schur over the time interval $[0, T]$.

As an aside, we would like to point out two issues. First, the necessary and sufficient exponential stability condition of this chapter extends easily to discrete repetitive processes under similar assumptions. As a matter of fact, the analysis can be more or less followed in the same manner, since the time domain dynamics were used insofar as proving boundedness/Lipschitz/asymptotic properties of the pass to pass operator. Second, the asymptotics (6.12) of the nonlinear perturbations show that it is indeed possible to find a similar result for $\mathcal{L}_2$ exponential stability[28]. When a system is $\mathcal{L}_\infty$ stable, we can guarantee that an $\mathcal{L}_2$ analogue of Lemma 6.2 holds for small (in $\mathcal{L}_\infty$) boundaries. Then, because $\mathcal{L}_\infty$ stability implies $\mathcal{L}_p$ stability in the linear case (Remark 6.2), we can go through similar analysis as before to reach the desired result.

To our knowledge, the work presented here is the first systematic study of stability for nonlinear repetive processes. The findings of the chapter are especially important since local stability is the precursor to global stability. The comprehensiveness of these results are reflected in the fact that they tie in the various existing results from nonlinear ILC analysis via a single framework. We hope that the analysis presented in the chapter will pave the way for further research on nonlinear repetitive processes, such as extensions to different classes of systems and the corresponding control strategies.

[28]Convergence is obvious since $\|.\|_{\mathcal{L}_2} \leq T\sqrt{m}\|.\|_{\mathcal{L}_\infty}$. Instead, we are interested in a bound of the form (6.4).

CHAPTER 7

Conclusion

In this dissertation, we laid out foundational material for classical ILC and pointed out research directions to relax the fundamental invariance assumption and to come up with a systematic and practical framework to encourage widespread use of ILC in experimental settings. In particular, in Chapter 4, we investigated the use of $\mathcal{L}_1$ AC in an ILC setting and how it enabled us to utilize an aggressive ILC design for fast convergence, all the while maintaining monotonic behavior in the time-iteration domains under time varying uncertainties and disturbances. While we assumed iteration invariance for the analysis, we saw in simulations that the combined $\mathcal{L}_1$-ILC scheme exhibited stable behavior with comparatively small transients. This encouraged us to investigate the basic stability analysis of iteration varying systems, which has not been extensively studied in the literature. Our technical analysis, simulation studies, and experimental results in Chapter 5 show that iteration varying systems are guaranteed to be stable when ILC update laws are designed to be robust, which can be done using existing methods from the literature. Further, when uncertainties converge in the iteration axis, convergence to a nominal system can be guaranteed, a result demonstrating the power of ILC over standard feedforward synthesis. As a next step to the signal space approach adopted in the analysis of iteration varying systems, we shifted the focus of our work to repetitive processes in Chapter 6, and showed that the exponential stability of a nonlinear system, its linearization, and the uniform Schur stability of the relevant linear state space matrix are equivalent. We utilized this result to analyze local stability of Picard iterations with nonconstant initial states, as well as nonlinear ILC algorithms. Our findings were supported by simulation studies on the model of an actuated Van der Pol oscillator with time varying damping, where it was shown that an ILC algorithm using the second derivative of the error can asymptotically find an input signal to uniformly track a sinusoidal reference without any stabilizing feedback.

The work presented in Chapters 4 to 6 is partially based on the following publications:

- B. Altin and K. Barton."Exponential Stability of Nonlinear Differential Repetitive Processes with Applications to Iterative Learning Control". In: *Automatica* (2016), revised and resubmitted.

- B. Altin, J. Willems, T. Oomen, and K. Barton. "Iterative Learning Control of Iteration Varying Systems via Robust Update Laws with Experimental Implementation". In: *Control Engineering Practice* (2016), under review.

- B. Altin and K. Barton. "On Linearized Stability of Differential Repetitive Processes and Iterative Learning Control". In: *Decision and Control (CDC), 2015 IEEE 54th Annual Conference on.* Dec. 2015, pp. 6064-6069.

- B. Altin and K. Barton. "Learning Control of Linear Iteration Varying Systems with Varying References through Robust Invariant Update Laws". In: *American Control Conference (ACC), 2015.* June 2015, pp. 4880-4885.

- B. Altin and K. Barton, "Robust iterative learning for high precision motion control through $\mathcal{L}_1$ adaptive feedback," in *Mechatronics*, vol.24, no. 6, pp. 549-561, 2014.

- B. Altin and K. Barton, "$\mathcal{L}_1$ adaptive control in an iterative learning control framework: Stability, robustness and design trade-offs," in *American Control Conference (ACC), 2013*, 2013, pp. 6697-6702.

- B. Altin and K. Barton, "$\mathcal{L}_1$ adaptive control in an iterative learning control framework for precision nanopositioning," in *Proc. of the ASPE Spring Top. Meet.*, vol. 55, 2013, pp. 88-93.

It is argued in an Automatica paper [18] that "Since a non-causal approach is the only viable route for ILC, future work should investigate the benefits of non-causal ILC versus conventional feedback control,". While we agree that the ability to consider noncausality is a strong aspect, ILC is a highly valuable technique that complements the limitations of traditional feedback such as finite bandwidth even in the causal case. Indeed, real life solutions are in essence a combination of different methodologies and seek to exploit the best of different worlds, and the results of Chapter 5 make a strong case for the value of ILC for iteration varying systems. In this sense, it would be beneficial for the ILC community to consider ILC as per the common interpretation of feedback control in the iteration domain, and expand the scope of ILC research towards repetitive processes and multidimensional systems in general, as discussed in Chapter 6.

While there are still many interesting questions specific to ILC and the reference tracking problem, it is our belief that repetitive processes and multidimensional systems will provide the generality needed to broaden the impact of ILC research. Indeed, it was shown in Chapter 6 that the Picard iterations, a method of utmost importance in the study of differential equations, can be readily modeled and analyzed as a nonlinear time varying repetitive process. On the application side, development of advanced control algorithms for nonlinear systems would open the door for the repetitive process control paradigm to be implemented on a variety of physical systems, with guaranteed performance and robustness bounds, and could revolutionize AM technology by significantly improving throughput and precision.

As mentioned before, the existing literature on repetitive processes are predominantly on linear systems, and hence there is a plethora of research directions that can be pursued to further our understanding of such systems. Extension of the theory to different classes of systems, for example, delay or partial differential equations (i.e. multidimensional in layer dynamics) are interesting problems that can have significant impact. More immediate questions that should be answered include estimation of the region of attraction, derivation of global stability conditions, and computation of finite dimensional Lyapunov functions, along with their counterparts for trial (iteration) varying systems. On the other hand, it is also important for researchers to apply the theoretical results in simulation or experiments to physical systems to push the boundaries of this relatively new control paradigm.

APPENDIX A

Supplemental Material for Chapter 4

The following sections lay out supplemental material related to chapter 4. The notation is consistent with that of chapter 4 unless otherwise stated.

A.1 Intermediate Technical Results

In this section we present some results that are helpful towards evaluating system uncertainties and establishing the relationship of Lemma 4.2. These will be used to show the existence and stability of the feedback operators. In the following discussion, $\mathcal{F}_1 : V_1 \to V_2$, $\mathcal{F}_2 : V_2 \to V_1$ are operators, where V_1 and V_2 are vector space.

We first give a generalization of the $\mathcal{L}_1$ norm condition which ensures that the objective of the $\mathcal{L}_1$ AC problem is well defined. While we assume that the inverse exists, the argument is valid regardless of its existence if we think of it as a feedback interconnection. Note that for linear systems, the condition $\|\mathcal{F}_2\mathcal{F}_1\| \leq \phi_2 < 1$ guarantees the existence of the inverse shown in Lemma 2.1. This is a special case of the small gain theorem [44, page 218].

Lemma A.1. *Assume V_1, V_2 to be endowed with norms such that $\|\mathcal{F}_1\| \leq \phi_1 < \infty$ and $\|\mathcal{F}_2\mathcal{F}_1\| \leq \phi_2 < 1$. Then, $\|\mathcal{F}_1(\mathcal{I} + \mathcal{F}_2\mathcal{F}_1)^{-1}\| \leq \phi_1(1-\phi_2)^{-1}$.*

Proof. Let $\zeta, \xi \in V$ be the input and output vectors of $\mathcal{I} + \mathcal{F}_2\mathcal{F}_1$; respectively. Since $\|\mathcal{F}_2\mathcal{F}_1\| \leq \phi_2 < 1$, we have $\|\xi\| \geq \|\zeta\| - \|\mathcal{F}_2\mathcal{F}_1\|\|\zeta\| \geq (1-\phi_2)\|\zeta\|$ by the reverse triangle inequality. Hence, $\|(\mathcal{I}+\mathcal{F}_2\mathcal{F}_1)^{-1}\| \leq (1-\phi_2)^{-1}$. The result then follows by submultiplicativity. ■

Let $F_1 \in \mathbb{C}^{m\times n}$, $F_2 \in \mathbb{C}^{n\times m}$. A generalization of the identity

$$(I + F_1F_2)^{-1} = I - F_1(I + F_2F_1)^{-1}F_2$$

to linear operators is given as follows.

Lemma A.2. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be linear operators. Then, $(\mathcal{I}+\mathcal{F}_1\mathcal{F}_2)^{-1}$ exists if and only if $(\mathcal{I}+\mathcal{F}_2\mathcal{F}_1)^{-1}$ exists. Moreover,*

$$(\mathcal{I}+\mathcal{F}_1\mathcal{F}_2)^{-1} = \mathcal{I} - \mathcal{F}_1(\mathcal{I}+\mathcal{F}_2\mathcal{F}_1)^{-1}\mathcal{F}_2,$$
$$(\mathcal{I}+\mathcal{F}_2\mathcal{F}_1)^{-1} = \mathcal{I} - \mathcal{F}_2(\mathcal{I}+\mathcal{F}_1\mathcal{F}_2)^{-1}\mathcal{F}_1.$$

Proof. Assume $\mathcal{I}+\mathcal{F}_1\mathcal{F}_2$ is invertible. Then, $(\mathcal{I}+\mathcal{F}_2\mathcal{F}_1)(\mathcal{I}-\mathcal{F}_2(\mathcal{I}+\mathcal{F}_1\mathcal{F}_2)^{-1}\mathcal{F}_1) = \mathcal{I}$ by direct computation, which shows that $\mathcal{I}-\mathcal{F}_2(\mathcal{I}+\mathcal{F}_1\mathcal{F}_2)^{-1}\mathcal{F}_1$ is an inverse of $\mathcal{I}+\mathcal{F}_2\mathcal{F}_1$. By interchanging $\mathcal{F}_1$ and $\mathcal{F}_2$, we can show the converse statement, thus completing the proof. ■

The following shows that the $\mathcal{L}_1$ norm of a system bounds its induced $\mathcal{L}_p$ norm.

Lemma A.3. *[44, page 200] Let $F(s)$ be a stable causal SISO LTI system. Then for every input signal $\zeta \in \mathcal{L}_{pe}$, $p \in [1,\infty]$, the output $\xi \in \mathcal{L}_{pe}$ and we have*

$$\|\xi_\tau\|_{\mathcal{L}_p} \leq \|F(s)\|_{\mathcal{L}_1}\|\zeta_\tau\|_{\mathcal{L}_p}.$$

A.2 Proofs of the Main Results

Proof of Lemma 4.2. Let $\zeta \in \mathcal{L}_{2e}^m$, $\xi \in \mathcal{L}_{2e}^n$ be the input and output signals; respectively. Then, by Lemma A.3, $\|(\xi_k)_\tau\|_{\mathcal{L}_2} = \|(\sum_{l=1}^m f_{kl} * \zeta_l)_\tau\|_{\mathcal{L}_2} \leq$

$\sum_{l=1}^{m} \|f_{kl}\|_{\mathcal{L}_1} \|(\zeta_l)_\tau\|_{\mathcal{L}_2}$, where $*$ denotes convolution. Let

$$\delta_k \triangleq \begin{bmatrix} \|f_{k1}\|_{\mathcal{L}_1} & \|f_{k2}\|_{\mathcal{L}_1} & \dots & \|f_{km}\|_{\mathcal{L}_1} \end{bmatrix}^T,$$
$$\epsilon \triangleq \begin{bmatrix} \|(\zeta_1)_\tau\|_{\mathcal{L}_2} & \|(\zeta_2)_\tau\|_{\mathcal{L}_2} & \dots & \|(\zeta_m)_\tau\|_{\mathcal{L}_2} \end{bmatrix}^T,$$

so by the Cauchy-Schwarz inequality

$$\sum_{l=1}^{m} \|(f_{kl})_\tau\|_{\mathcal{L}_1} \|(\zeta_l)_\tau\|_{\mathcal{L}_2} = \delta_k^T \epsilon \leq \|\delta_k\|_2 \|\epsilon\|_2 \leq \|\delta_k\|_1 \|\epsilon\|_2.$$

Moreover, $\|\delta_k\|_1 \leq \|F(s)\|_{\mathcal{L}_1}$ and $\|\epsilon\|_2 = \|\zeta_\tau\|_{\mathcal{L}_2}$ by definition, which imply

$$\|(\xi_k)_\tau\|_{\mathcal{L}_2} \leq \|F(s)\|_{\mathcal{L}_1} \|\zeta_\tau\|_{\mathcal{L}_2}.$$

Thus, $\|\xi_\tau\|_{\mathcal{L}_2} \leq \sqrt{n} \|F(s)\|_{\mathcal{L}_1} \|\zeta_\tau\|_{\mathcal{L}_2}$. Then by Theorem 4.1, $\|F(s)\|_\infty \leq \sqrt{n} \|F(s)\|_{\mathcal{L}_1}$.

■

Proof of Theorem 4.4. By the Cauchy-Schwarz inequality,

$$|\theta^T G(j\omega)| \leq \|\theta\|_2 \|G(j\omega)\|_2 \leq \theta_{M_2} \|G(s)\|_\infty, \quad \forall \omega \in \mathbb{R}.$$

Note that $\theta_{M_2} = \sqrt{n}\theta_M$, so by Lemma 4.2, we have

$$\|\theta^T G(s)\|_\infty \leq \theta_{M_2} \|G(s)\|_\infty \leq \theta_{M_1} \|G(s)\|_{\mathcal{L}_1} < 1.$$

Let $G_\theta(s) \triangleq \frac{\theta^T G(s)}{1-\theta^T G(s)}$, which implies $\bar{H}(s) = H(s)(1 + G_\theta(s))$. By Lemma A.1,

$$|G_\theta(j\omega)| \leq \|G_\theta(s)\|_\infty \leq \frac{\theta_{M_2} \|G(s)\|_\infty}{1 - \theta_{M_2} \|G(s)\|_\infty} = \kappa, \quad \forall \omega \in \mathbb{R}.$$

Assume there exists $\mu \in [0, 1)$ such that

$$|Q(j\omega)(1 - L(j\omega)\bar{H}(j\omega))| \leq$$
$$|Q(j\omega)||(1 - L(j\omega)H(j\omega))| + \kappa|Q(j\omega)||L(j\omega)||H(j\omega)| \leq \mu,$$

for all $\omega \in \mathbb{R}$. But then, this is equivalent to (4.9). ■

Proof of Lemma 4.4. We extend $\mathcal{L}_1$ to include complex transfer functions and note that the inverse Laplace transform of $1/(s + p)$ is $e^{-pt}1(t)$, which implies

$$\left\| \frac{1}{s+p} \right\|_{\mathcal{L}_1} = \frac{1}{\Re(p)}, \quad \forall p : \Re(p) > 0.$$

Without loss of generality, assume $m = n - 1$. Then we have,

$$\|F(s)\|_{\mathcal{L}_1} \leq \left\| \frac{1}{s+p_n} \right\|_{\mathcal{L}_1} \prod_{k=1}^{n-1} \left\| \frac{s+z_k}{s+p_k} \right\|_{\mathcal{L}_1}$$
$$\leq \frac{1}{\Re(p_n)} \prod_{k=1}^{n-1} 1 + \frac{|z_k|}{\Re(p_k)} + \frac{|p_k|}{\Re(p_k)}.$$

Now the assumption $\arg(p_k) \in [\psi, 2\pi - \psi]$ implies that $1/\Re(p_k) \to 0$ as $|p_k| \to \infty$ and $|p_k|/\Re(p_k)$ is bounded for $p_k \neq 0$. It follows that $1 + |z_k|/\Re(p_k) + |p_k|/\Re(p_k)$ is $O(1)$ for $k = 1, 2, \ldots, n - 1$. Since $1/\Re(p_n) \to 0$, the result follows. ■

Proof of Theorem 4.7. The proof follows the same ideas of Theorem 4.4. We first show that the $\mathcal{L}_2$ gain of $\mathcal{W}\mathcal{G}_m$ is less than 1 due to the $\mathcal{L}_1$ norm condition. Note that $\mathcal{W}$ is made up of 3 cascaded systems (4.16) with the first one being LTI. The readers can therefore easily verify $\Psi \triangleq M_2\|T(s)A_TG_m(s)\|_\infty$ to be an upper bound on the induced $\mathcal{L}_2$ norm of $\mathcal{W}\mathcal{G}_m$. From Lemma 4.2 and the equality $\theta_{M_1} = \sqrt{n}\theta_{M_2}$ it follows that $\Psi \leq M\|G_m(s)\|_{\mathcal{L}_1} < 1$. Since the $\mathcal{L}_2$ gain of $\mathcal{W}\mathcal{G}_m$ is less than 1, it follows that $\mathcal{I} - \mathcal{W}\mathcal{G}_m$ is invertible with $\mathcal{L}_2$ gain less than $1 - \Psi$ by Lemmas 2.1 and A.1.

Let $\mathcal{G}_{m\theta} \triangleq (\mathcal{I}-\mathcal{C})(\mathcal{I}-\mathcal{W}\mathcal{G}_m)^{-1}\mathcal{W}\mathcal{H}_{xm}$, where $\mathcal{H}_{xm}$ is $H_{xm}(s)$ in operator notation. It follows that an upper bound on the $\mathcal{L}_2$ gain of $\mathcal{G}_{m\theta}$ is

$$\frac{M_2\|T(s)A_T H_{xm}(s)\|_\infty \|1-C(s)\|_\infty}{1-M_2\|T(s)A_T G_m(s)\|_\infty},$$

which is equal to $\kappa_{\mathrm{m_{tv}}}$ by definition. Moreover, by Lemma A.2, the mapping $\bar{\mathcal{H}}_m$ from v_i to y_i is given by $\bar{\mathcal{H}}_m = \mathcal{H}_m(\mathcal{I}+\mathcal{G}_{m\theta})$. Now let $\mathcal{Q}$, $\mathcal{L}$ be $Q(s)$ and $L(s)$ in operator notation; respectively. Assume there exists $\mu_{m_{\mathrm{tv}}} \in [0,1)$ such that,

$$\begin{aligned}\|\mathcal{Q}(\mathcal{I}-\mathcal{L}\bar{\mathcal{H}}_m)\|_{\mathcal{L}_2} &\leq \|\mathcal{Q}(\mathcal{I}-\mathcal{L}\mathcal{H}_m)\|_{\mathcal{L}_2} + \kappa_{m_{\mathrm{tv}}}\|\mathcal{Q}\mathcal{L}\mathcal{H}_m\|_{\mathcal{L}_2} \\ &\leq \mu_{m_{\mathrm{tv}}}.\end{aligned}$$

But then, this is equivalent to (4.29) by Theorem 4.1. ∎

Remark A.1. The existence of $(\mathcal{I}-\mathcal{W}\mathcal{G}_m)^{-1}$ can also be proven by $M\|G_m(s)\|_{\mathcal{L}_1} < 1$ or any norm that satisfies the small gain condition. This property would be useful if it cannot be shown that the $\mathcal{L}_2$ gain is less than the $\mathcal{L}_1$ norm. For instance, if the inequality $\|F(s)\|_{\mathcal{L}_1} < 1$ is true, but $\|F(s)\|_\infty < 1$ is not necessarily true,

$$\|(I-F(s))^{-1}\|_\infty \leq \sqrt{n}/(1-\|F(s)\|_{\mathcal{L}_1}),$$

by Lemmas 4.2 and A.1. Obviously, this would lead to a more restrictive robust convergence condition.

A.3 $\mathcal{L}_1$ Adaptive Output Feedback Controller Definitions

We list the variables that are used in Section 4.4 below. The readers can refer to [117] for the original definitions, we provide several modifications to account for the addition of $v_i(t)$ in the adaptive controller.

$$\rho_1 \triangleq \frac{|k_g|\|H_{xm}(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H_{xm}(s)\|_{\mathcal{L}_1}\|v_i\|_{\mathcal{L}_\infty} + \|G_m(s)\|_{\mathcal{L}_1}(\|\sigma_m\|_{\mathcal{L}_\infty} + M\rho_2)}{1 - \|G_m(s)\|_{\mathcal{L}_1}M},$$

where $\rho_2 \triangleq \|x_{\text{ref}_2}\|_{\mathcal{L}_\infty}$; and $x_{\text{ref}_2}(t)$ is defined according to

$$\dot{x}_{\text{ref}_2}(t) = A_m x_{\text{ref}_2}(t), \quad x_{\text{ref}_2}(0) = \hat{x}_{\text{in}}.$$

Let $\rho \triangleq \rho_1 + \rho_2$ and

$$\bar{\Delta} \triangleq \Delta_m + M\left(\rho + \bar{\gamma}\frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G_m(s)\|_{\mathcal{L}_1}M}\right),$$

where $\bar{\gamma} > 0$ is arbitrary. Let

$$\begin{aligned} \beta_1 &\triangleq \beta_{01}\tfrac{\|C(s)\|_{\mathcal{L}_1}}{1-\|G_m(s)\|_{\mathcal{L}_1}M}, & \beta_2 &\triangleq \beta_{02} + \beta_{01}\rho, \\ \beta_3 &\triangleq \tfrac{\lambda_{\max}(P_m)}{\lambda_{\min}(Z_m)}\beta_1, & \beta_4 &\triangleq 4\bar{\Delta}^2 + \tfrac{\lambda_{\max}(P_m)}{\lambda_{\min}(Z_m)}\beta_2, \end{aligned}$$

where

$$\begin{aligned} \beta_{01} &\triangleq 4\bar{\Delta}M\left(d_\theta/\theta_{M_1} + \|A_m\|_{\mathcal{L}_1} + \|b_m\|_{\mathcal{L}_1}M\right), \\ \beta_{02} &\triangleq 4\bar{\Delta}\left(d_{\sigma_m} + M\|b_m\|_{\mathcal{L}_1}\left(\|C(s)\|_{\mathcal{L}_1}\left(|k_g|\|r\|_{\mathcal{L}_\infty} + \bar{\Delta}\right) + \|v_i\|_{\mathcal{L}_\infty} + \Delta_m\right)\right). \end{aligned}$$

The transient bounds of the controller are given by

$$
\begin{aligned}
\gamma_0 &\triangleq \sqrt{\frac{\alpha\beta_4}{\Gamma_c \lambda_{\min}(P_m)}} \\
\gamma_1 &\triangleq \gamma_0 \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G_m(s)\|_{\mathcal{L}_1} M}, \\
\gamma_2 &\triangleq \gamma_1 M \|C(s)\|_{\mathcal{L}_1} + \gamma_0 \left\| \frac{C(s)}{c_o^T H_{xm}(s)} c_o^T \right\|_{\mathcal{L}_1},
\end{aligned}
$$

where $c_o \in \mathbb{R}^n$ is arbitrary such that $c_o^T H_{xm}(s)$ is minimum phase and has relative degree 1.

A.4 Criticisms of $\mathcal{L}_1$ Adaptive Control

$\mathcal{L}_1$ AC theory has received significant criticism from the adaptive control community despite successful implementations on many practical systems, including safety critical flight control systems [149]. To date, several prominent members of the adaptive control community have expressed their doubts towards the theory in several publications, such as [150–153]. Reference [150] is of particular importance, where it was noted that $\mathcal{L}_1$ AC offers "no benefits in terms of robustness, performance, or bounds that suggest useful trade-offs", and raises the following issues about the theory:

1. The insertion of the input filter deteriorates tracking performance.

2. The use of an input filter deteriorates stability margins.

3. The recommended use of high adaptive gains has a negative effect on robustness, and causes numerical instability.

In response to some of the claims raised in an earlier version of [150], a document clarifying the main properties of $\mathcal{L}_1$ AC [154] has been published online by Naira Hovakimyan. Based on our reading of [150, 154] and other material on $\mathcal{L}_1$ AC theory, we respond to the above claims as follows:

1. It is true that the input filter deteriorates tracking performance in terms of the MRAC "desired system". As a matter of fact the bounds establishing tracking performance in $\mathcal{L}_1$ AC are stated in terms of the uncertain "reference model", which describes the performance-robustness trade-off and helps bridge the gap between adaptive control and robust control. However, it is correctly pointed out in [150] that the control objective stated in [155], "to design an adaptive controller to ensure that the system output $y(t)$ follows a given reference signal $r(t)$ with quantifiable transient and steady-state performance bounds", is not achievable. It seems that this is a misstatement on the authors' part, since the objective of MRAC and therefore $\mathcal{L}_1$ AC is (partial) model following, not reference tracking.

2. The claim in [150] that filtering deteriorates robustness margins seem to be due to the fact that the considered controller has a filtered MRAC structure, which corresponds to the simplest $\mathcal{L}_1$ AC architecture, that of systems with unknown pole locations. While is true that the robustness properties of conventional MRAC for unknown pole locations is deteriorated by filtering, the correct $\mathcal{L}_1$ AC architecture for the unmodeled dynamics scenario in [150] is presented in [154], where parametric uncertainty in the input gain is considered. It is shown in [110,154] through a theoretical proof and numerical simulations that for this architecture, as opposed to conventional MRAC, the time delay margins of $\mathcal{L}_1$ AC are uniformly bounded away from zero, independent of the estimation gain. A secondary claim made by the authors of [150] is that the $\mathcal{L}_1$ AC architecture for input uncertain systems requires the knowledge of unmodeled dynamics for the design of its filtering structure, as seen in equation (2.130) of [110]. However, upon careful reading, it can be observed that the filter defined in equation (2.130) of [110] is for analysis purposes only, and is not used in the implementation of the controller.

3. As the authors of [150] mention, the use of high adaptive gains may cause numerical issues and can lead to chatter, as shown in [150, 154]. Nevertheless, as a result of the filtering structure, the high frequency content of the estimate does not propagate to the control channel. Moreover, since the time delay margins of $\mathcal{L}_1$ AC are uniformly bounded away from zero, the adaptation gain can be safely increased to track the reference model arbitrarily closely, thereby increasing the predictability (and hence "robustness", in some sense) of the system. In experimental implementations, this adaptation rate is bounded above by practical concerns, as detailed in [110, 149] through flight control systems.

Another criticism towards the theory appears in [152, 153], where it is suggested that there exist scalar LTI systems and reference models such that the $\mathcal{L}_1$ stability condition cannot be be satisfied, under the assumption that the bandwidth of the filter should be lower than that of the reference model. It is argued in an online document [156] that this assumption is not justified, on the grounds that "the filter acts as an additional actuator", and "if its dynamics are slower than the plant, this will limit both the performance and the robustness of the closed-loop system". Hence, the stability condition can be satisfied by selecting a sufficiently high bandwidth filter.

Finally, it has been recently suggested that "adaptation is unnecessary" in $\mathcal{L}_1$ AC in some sense, since $\mathcal{L}_1$ AC approximates linear controllers: It is shown in [151–153] that for a simple class of $\mathcal{L}_1$ AC algorithms, the control signal converges to that of a proportional-integral controller as the perturbation term due to adaptation tends to zero. This controller is *implementable* since it does not require the knowledge of the unknown parameters. Building on these works, [157] analyzes the issue of implementability for parametrized linear controllers in state space form. It is also interesting to note that this issue is studied in detail in [158] in a different fashion, where it is shown by algebraic manipulations that for a number of cases the limiting $\mathcal{L}_1$

controllers[29] are indeed implementable under certain conditions, and often resemble known architectures like disturbance observers. Hence, by increasing the adaptation gain, the system approximates an implementable closed loop arbitrarily closely. For example, consider the reference model of Section 4.3 in the Laplace transform domain, extended to time varying uncertainties:

$$\begin{aligned} x_{\mathrm{ref}}(s) &= H_x(s)(u_{\mathrm{ref}}(s) + \eta_{\mathrm{ref}}(s)), \\ y_{\mathrm{ref}}(s) &= H(s)(u_{\mathrm{ref}}(s) + \eta_{\mathrm{ref}}(s)), \\ u_{\mathrm{ref}}(s) &= C(s)(k_g r(s) - \eta_{\mathrm{ref}}), \end{aligned}$$

where $\eta_{\mathrm{ref}}(s)$ is the Laplace transform of $\theta^T(t)x_{\mathrm{ref}}(t)$, and the feedforward input is omitted. Then, since $\eta_{\mathrm{ref}}(s) = y_{\mathrm{ref}}(s)/H(s) - u_{\mathrm{ref}}(s)$, it follows by simple manipulations that

$$u_{\mathrm{ref}}(s) = \frac{C(s)}{1-C(s)}\left(k_g r(s) - \frac{y_{\mathrm{ref}}(s)}{H(s)}\right) = \frac{C(s)}{1-C(s)}k_g r(s) - \frac{C(s)}{(1-C(s))H(s)}y_{\mathrm{ref}}(s).$$

Also observe that an equivalent state feedback controller can be found in a similar manner by considering a left inverse $H_x^L(s)$ of $H_x(s)$:

$$\begin{aligned} u_{\mathrm{ref}}(s) &= \frac{C(s)}{1-C(s)}\left(k_g r(s) - H_x^L(s)x_{\mathrm{ref}}(s)\right) \\ &= \frac{C(s)}{1-C(s)}k_g r(s) - \frac{C(s)}{1-C(s)}H_x^L(s)x_{\mathrm{ref}}(s). \end{aligned}$$

Note that for the above output feedback control law to be realizable, the relative degree of $C(s)$ should be greater than or equal to that of $H(s)$. As such, the limiting controller for the $\mathcal{L}_1$ architecture may not always be implementable for systems with relative degree greater than 1: For the $\mathcal{L}_1$ controller of Section 4.5, the equivalent controller is not implementable since the relative degree of the plant is 3, while the

[29] That is, the reference model controllers.

relative degree of $C(s)$ is 1. Hence, in order to consider an $\mathcal{L}_1$ equivalent LTI controller, we would have to limit our attention to filters with relative degree of at least 3. Whether a lower relative degree has structural benefits from a performance-robustness standpoint is an open question; indeed, it has been pointed out that the constrained optimal design problem of the filter is nonconvex, and hard to address [110].

It is noted in [158] that while the limiting controllers make explicit use of the system inverse, the adaptive architectures approximate the inverse, and this property is essential in extending the methodology to nonlinear systems and also accommodating various known hardware constraints like saturation and delay [159]. In [160], linear state feedback $\mathcal{L}_1$ AC is analyzed and it is shown that as the adaptive gain goes to infinity, the limiting controller is recovered in a local sense, regardless of unmodeled dynamics and signals. Whether this property holds in a large region of operation[30], or with more complex $\mathcal{L}_1$ AC architectures is an open question. As such, $\mathcal{L}_1$ architectures may be useful in the compensation of unmodeled dynamics and exogenous signals. In addition, the study of the limiting behavior of different classes of $\mathcal{L}_1$ AC architectures may aid in the synthesis of robust, high performance controllers for highly uncertain systems [159, 160]. These problems require further attention.

[30]It is shown in [160] that despite the algebraic equivalence, the $\mathcal{L}_1$ controller and the linear controller have different disturbance rejection characteristics

APPENDIX B

Supplemental Material for Chapter 6

The following sections lay out supplemental material related to chapter 6. The notation is consistent with that of chapter 6 unless otherwise stated.

B.1 Proofs of Technical Results

Proof of Lemma 6.1. We begin by defining the set

$$\bar{\mathcal{Y}} \triangleq \{u \in \mathcal{Y} : u(t) \in Y, \forall t \in [0,T]\},$$

and note that for any $u \in \bar{\mathcal{Y}}$, $\bar{f}(\xi, t) \triangleq f(\xi, u(t), t)$ is continuous in $t \in [0,T]$ for all $\xi \in X$ since f is continuous in Z. Moreover, as f is continuously differentiable, it is also Lipschitz on the compact set Z. That is, there exists a constant L_f such that

$$\|f(\xi_1, \upsilon_1, \tau_1) - f(\xi_2, \upsilon_2, \tau_2)\| \leq L_f \|(\xi_1, \upsilon_1, \tau_1) - (\xi_2, \upsilon_2, \tau_2)\|,$$

for all $(\xi_1, \upsilon_1, \tau_1)$ and $(\xi_2, \upsilon_2, \tau_2)$ in Z. In turn, this implies that

$$\|\bar{f}(\xi_1, t) - \bar{f}(\xi_2, t)\| \leq L_f \|\xi_1 - \xi_2\|,$$

for all $\xi_1, \xi_2 \in X$ and $t \in [0,T]$, *and* any $u \in \bar{\mathcal{Y}}$, so $\bar{f}(\xi, t)$ is Lipschitz with respect to ξ, uniformly over time and the space of inputs. Now consider $\dot{\chi}_i(t) = f(\chi_i(t), u_i(t), t)$, where the initial conditions and inputs satisfy the inequality $\|\chi_i(0)\| + \|u_i\|_{\mathcal{L}_\infty} < \delta$, for all $i \in \{1,2\}$. By Assumption 6.1, the integral curves of both systems reside in X. In addition, $\bar{f}_i(\xi,t) \triangleq f(\xi,u_i(t),t)$ is continuous in $t \in [0,T]$ for all $\xi \in X$, and Lipschitz with respect to ξ on $X \times [0,T]$, for all $i \in \{1,2\}$. Define the function $\tilde{f}(\xi,t) \triangleq \bar{f}_2(\xi,t) - \bar{f}_1(\xi,t)$, and rewrite the two systems as

$$\begin{aligned} \dot{\chi}_1(t) &= \bar{f}_1(\chi_1(t),t), \\ \dot{\chi}_1(t) &= \bar{f}_1(\chi_2(t),t) + \tilde{f}(\chi_2,t). \end{aligned} \tag{B.1}$$

Since f is Lipschitz on Z, as in the previous case where we showed that $\bar{f}$ is Lipschitz with respect to its first argument, it follows that $\|\tilde{f}(\xi,t)\| \leq L_f \|u_1(t) - u_2(t)\|$ for all $(\xi,t) \in X \times [0,T]$. As $u_1, u_2 \in \bar{Y}$, this also means that

$$\|\tilde{f}(\xi,t)\| \leq L_f \|u_1 - u_2\|_{\mathcal{L}_\infty} < M,$$

for some M, for all $(\xi,t) \in X \times [0,T]$ and $u_1, u_2 \in \bar{\mathfrak{Y}}$ since Y is compact. Now, (B.1) satisfies all assumptions of theorem 3.4 of [44], which states that

$$\|\chi_1(t) - \chi_2(t)\| \leq \|\chi_1(0) - \chi_2(0)\| e^{L_f t} + \|u_1 - u_2\|_{\mathcal{L}_\infty}(e^{L_f t} - 1), \quad \forall t \in [0,T],$$

and therefore

$$\|\chi_1 - \chi_2\|_{\mathcal{L}_\infty} \leq L_1(\|\chi_1(0) - \chi_2(0)\| + \|u_1 - u_2\|_{\mathcal{L}_\infty}),$$

where the Lipschitz constant $L_1 = e^{L_f T}$. ∎

Proof of Corollary 6.1. Since g is continuously differentiable, it is also Lipschitz in Z. In other words, there exists L_g such that

$$\|g(\xi_1,\upsilon_1,\tau_1) - g(\xi_2,\upsilon_2,\tau_2)\| \leq L_g \|(\xi_1,\upsilon_1,\tau_1) - (\xi_2,\upsilon_2,\tau_2)\|,$$

for all $(\xi_1, \upsilon_1, \tau_1)$ and $(\xi_2, \upsilon_2, \tau_2)$ in Z. Thus, given state-input vector pairs (χ_i, u_i) and the corresponding output vectors w_i, for all $i \in \{1, 2\}$,

$$\|w_1 - w_2\|_{\mathcal{L}_\infty} \leq L_g(\|\chi_1 - \chi_2\|_{\mathcal{L}_\infty} + \|u_1 - u_2\|_{\mathcal{L}_\infty}),$$

if $(\chi_i(t), u_i(t)) \in \mathrm{cl}(X) \times Y$ for all $t \in [0, T]$. As the composition of Lipschitz maps is also Lipschitz, and the differential equation in (6.2) is Lipschitz within a δ ball around $0 \in \mathbb{R} \times \bar{Y}$ from Lemma 6.1, it follows that the input-output operator is also Lipschitz. ■

Proof of Lemma 6.2. By Lemma 6.1, since the Lipschitz constant $L_1 = e^{L_f T} > 1$, the following is true:

$$\|u\|_{\mathcal{L}_\infty} + \|\chi(0)\| < \delta, \implies \|(\chi, u)\|_{\mathcal{L}_\infty} \leq L_1(\|u\|_{\mathcal{L}_\infty} + \|\chi(0)\|).$$

Moreover, by (6.12), for any $\epsilon > 0$ there exists $\delta_O > 0$ such that $\|(\chi, u)\|_{\mathcal{L}_\infty} < \delta_O$ implies

$$\|\varphi(\chi, u)\|_{\mathcal{L}_\infty} < (\epsilon / L_1)\|(\chi, u)\|_{\mathcal{L}_\infty}.$$

Therefore, if $\|u\|_{\mathcal{L}_\infty} + \|\chi(0)\| < \delta^* < \min\{\delta, \delta_O / L_1\}$, it follows that

$$\|(\chi, u)\|_{\mathcal{L}_\infty} \leq L_1(\|u\|_{\mathcal{L}_\infty} + \|\chi(0)\|) < \delta_O,$$

and consequently

$$\|\varphi(\chi, u)\|_{\mathcal{L}_\infty} < (\epsilon / L_1)\|(\chi, u)\|_{\mathcal{L}_\infty} < \epsilon(\|u\|_{\mathcal{L}_\infty} + \|\chi(0)\|).$$

■

B.2 Discussion of Claim 6.3

As we have assumed no more than continuity of the state matrices for G_0, it will suffice to show that G_0 is bounded. Since B, C, D are all continuous, they are bounded on $[0, T]$, and it is a straightforward matter to show that the multiplication operators defined by these matrices are bounded with respect to any $\mathcal{L}_p$ norm, $p \in [1, \infty]$. Therefore it suffices to show that the time varying convolution operator defined by the corresponding state transition matrix is bounded. Because A is continuous, the state transition matrix Φ is continuously differentiable with respect to its first and second arguments on $[0, T]^2$ (see [146], page 62). As continuity of the partials imply differentiability, it follows that Φ is continuous and therefore bounded on $[0, T]^2$. Consequently, for any $i, j \in \{1, 2, \ldots, n\}$,

$$\sup_{t \in [0,T]} \int_0^t |\Phi_{ij}(t, \tau)| \, d\tau, \qquad \sup_{\tau \in [0,T]} \int_\tau^T |\Phi_{ij}(t, \tau)| \, dt,$$

are finite, where Φ_{ij} is the entry at the i-th row, j-th column of Φ. By theorem 75 of [161], page 306, it follows that the convolution operator is $\mathcal{L}_p$ stable for all $p \in [0, \infty]$.

Remark B.1. The bounded integral conditions for $\mathcal{L}_p$ stability given in [161] is modified here so that the supremum is taken over $t, \tau \in [0, T]$. This is because the state transition matrix Φ can be continuously extended to the first quadrant of $\mathbb{R}^2$ (the system is causal) that decays fast enough so the conditions hold over an infinite horizon.

B.3 Proof of Proposition 6.1

By (6.14), the output at pass $k+1$ can be written as

$$y_{k+1} = \bar{y}_{k+1} + \bar{G}_0(y_k - \bar{y}_k) + \Omega(\varphi(x_{k+1}, y_k)) = \bar{H}x_{k+1}(0) + \bar{G}_0 y_k + \Omega(\varphi(x_{k+1}, y_k))$$

so

$$y_k = \bar{G}_0^k y_0 + \sum_{i=1}^{k} \bar{G}_0^{k-i}(\bar{H}x_i(0) + \Omega(\varphi(x_i, y_{i-1}))) \tag{B.2}$$

for all $k \in \mathbb{N}$, when the solution exists. Recalling the fact that $\|\bar{G}_0^k\|_{\mathcal{L}_\infty} \leq \bar{M}\bar{\zeta}^k$ for all $k \in \mathbb{N}$ for some $\bar{M} \geq 1$ and $\bar{\zeta} \in (0,1)$, from (B.2), it follows that

$$\begin{aligned}\|y_N\|_{\mathcal{L}_\infty} \leq \bar{M}\bar{\zeta}^N \|y_0\|_{\mathcal{L}_\infty} + \max\left\{\|\bar{H}\|_{\mathcal{L}_\infty}, \|\Omega\|_{\mathcal{L}_\infty}\right\} \\ \times \left(\|\mathbf{x}(0)\|_{e_1} + \max_{i\in\{1,2,\ldots,N\}} \|\varphi(x_i, y_{i-1})\|_{\mathcal{L}_\infty}\right) \sum_{i=1}^{N} \bar{M}\bar{\zeta}^{N-i},\end{aligned}$$

therefore

$$\begin{aligned}\|y_N\|_{\mathcal{L}_\infty} \leq \underbrace{\bar{M}\bar{\zeta}^N}_{r_1<1} \|y_0\|_{\mathcal{L}_\infty} + \underbrace{\bar{M}\frac{1-\bar{\zeta}^N}{1-\bar{\zeta}} \max\{\|\bar{H}\|_{\mathcal{L}_\infty}, \|\Omega\|_{\mathcal{L}_\infty}\}}_{r_2>0} \\ \times \left(\|\mathbf{x}(0)\|_{e_1} + \max_{i\in\{1,2,\ldots,N\}} \|\varphi(x_i, y_{i-1})\|_{\mathcal{L}_\infty}\right). \end{aligned} \tag{B.3}$$

The rest of the proof will be divided into three steps:

B.3.1 Lyapunov Stability

This part follows the same basic ideas of lemma 3 of [41]. We take any

$$\epsilon \in \left(0, \frac{1-r_1}{r_2}\right),$$

where r_1 and r_2 are defined in (B.3). By Lemmas 6.2 and 6.3, since $L_f \geq 1$ there exist $\delta^*, \delta^*_{\text{fh}}$ satisfying

$$0 < \delta^*_{\text{fh}} < \min\{\delta_{\text{fh}}, \delta^*/L_{\text{fh}}\} \leq \delta^* \leq \epsilon,$$

such that $\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1} < \delta^*_{\text{fh}} < \min\{\delta_{\text{fh}}, \delta^*\}$ means

$$\|y_k\|_{\mathcal{L}_\infty} \leq L_{\text{fh}}(\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1}) < \delta^* \leq \epsilon, \tag{B.4}$$

which in turn implies

$$\|\varphi(x_k, y_{k-1})\|_{\mathcal{L}_\infty} < \epsilon/(L_{\text{fh}} + 1)(\|y_k\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1}) < \epsilon(\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1}),$$

for all $k \in \{1, 2, \ldots, N\}$. Assume $\|y_0\|_{\mathcal{L}_\infty} < \delta_y \leq \delta^*_{\text{fh}}/2$ and $\|\mathbf{x}(0)\|_{e_1} < \delta_x \leq r_y \delta_y$ for arbitrary r_y satisfying

$$r_y \in \left(0, \min\left\{1, \frac{1 - r_1 - r_2\epsilon}{r_2(1 + \epsilon)}\right\}\right).$$

The interval above is nonempty since $\epsilon < (1 - r_1)/r_2$, and if δ_x belongs to this interval, $\delta_x + \delta_y < 2\delta_y \leq \delta^*_{\text{fh}}$. It follows that,

$$\begin{aligned}\|y_N\|_{\mathcal{L}_\infty} &\leq r_1\|y_0\|_{\mathcal{L}_\infty} + r_2(\|\mathbf{x}(0)\|_{e_1} + \epsilon(\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_1})) \\ &\leq \|y_0\|_{\mathcal{L}_\infty}(r_1 + r_2\epsilon) + \|\mathbf{x}(0)\|_{e_1} r_2(1 + \epsilon),\end{aligned}$$

so $\|y_N\|_{\mathcal{L}_\infty} \leq \delta_y(r_1 + r_2\epsilon) + \delta_x r_2(1 + \epsilon) = r_N\delta_y < \delta_y$, where

$$r_N \triangleq (r_1 + r_2\epsilon) + r_y r_2(1 + \epsilon) < 1.$$

Moreover, by (B.4), $\|y_k\|_{\mathcal{L}_\infty} \le \epsilon$ for all $k \in \{1, 2, \dots, N-1\}$. By induction, it follows that $\|y_0\|_{\mathcal{L}_\infty} < \delta_y$ and $\|\mathbf{x}(0)\|_{e_1} < \delta_x$ implies $\|y_k\|_{\mathcal{L}_\infty} < \epsilon$ for all $k \in \mathbb{N}$, since $\delta_y < \epsilon$. Thus, if

$$\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\| < \delta_1 = \min\{\delta_x, \delta_y\},$$

then $\|y_k\|_{\mathcal{L}_\infty} < \epsilon$ for all $k \in \mathbb{N}$. As we can find such a $\delta_1 > 0$ for arbitrarily small $\epsilon > 0$, we conclude that the nonlinear system is stable.

B.3.2 Asymptotic Stability

From (B.2), we have

$$y_k = \bar{G}_0^k y_0 + \sum_{i=1}^{k} \bar{G}_0^{k-i}(\bar{H}x_i(0) + \Omega(\varphi(x_i, y_{i-1}))) = \bar{y}_k + \sum_{i=1}^{k} \bar{G}_0^{k-i}\Omega(\varphi(x_i, y_{i-1})).$$

Let $\epsilon = (1-\bar{\zeta})/(2\bar{M}\|\Omega\|_{\mathcal{L}_\infty})$. Since the system is stable, by Lemma 6.2 there exists a positive scalar δ_1 so that $\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)_{e_1}\| < \delta_2 = \delta_1$ implies

$$\begin{aligned}\limsup_{k\to\infty} \|y_k\|_{\mathcal{L}_\infty} &\le \epsilon\bar{M}\|\Omega\|_{\mathcal{L}_\infty} \limsup_{k\to\infty} \sum_{i=1}^{k} \bar{\zeta}^{k-i}(\|x_i(0)\| + \|y_{i-1}\|_{\mathcal{L}_\infty}) \\ &= \epsilon\bar{M}\|\Omega\|_{\mathcal{L}_\infty} \limsup_{k\to\infty} \underbrace{\sum_{i=1}^{k} \bar{\zeta}^{k-i}\|y_{i-1}\|_{\mathcal{L}_\infty}}_{\bar{S}_k}, \end{aligned} \tag{B.5}$$

as $\bar{y}_k \to 0$, and $\sum_{i=1}^{k} \bar{\zeta}^{k-i}\|x_i(0)\| \to 0$ if $\mathbf{x}(0) \in c_0$, as we have shown before in Section 6.4. Now, it is easy to verify that $\bar{S}_{k+1} = \bar{\zeta}\bar{S}_k + \|y_k\|_{\mathcal{L}_\infty}$, where $\bar{S}_k$ is defined in (B.5). Hence by (B.5) and Claim 6.1

$$\begin{aligned}\limsup_{k\to\infty} \|y_k\|_{\mathcal{L}_\infty} &\le \frac{\epsilon\bar{M}\|\Omega\|_{\mathcal{L}_\infty}}{1-\bar{\zeta}} \limsup_{k\to\infty} \|y_k\|_{\mathcal{L}_\infty} \\ &\le \frac{1}{2} \limsup_{k\to\infty} \|y_k\|_{\mathcal{L}_\infty},\end{aligned}$$

so $\limsup_{k\to\infty} \|y_k\|_{\mathcal{L}_\infty} = 0$, thus $\lim_{k\to\infty} \|y_k\|_{\mathcal{L}_\infty} = 0$. Therefore, the system is asymptotically stable.

B.3.3 Exponential Stability

Let $\mathbf{x}_\kappa(0) \triangleq \{x_{k+1}(0)\}_{k=\kappa}^{\infty}$ for any $\kappa \in \mathbb{N}$. As we have proved Lyapunov stability, given $\epsilon > 0$, by Lemma 6.2 and (B.3), we can find a constant $\delta_3 \in \{0, \delta_{\text{fh}}\}$ such that $\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda} < \delta_3$ implies

$$\begin{aligned}\|y_{(k+1)N}\|_{\mathcal{L}_\infty} &\le r_1\|y_{kN}\|_{\mathcal{L}_\infty} + r_2(\lambda^{kN}\|\mathbf{x}(0)\|_{e_\lambda} + \epsilon(\|y_{kN}\|_{\mathcal{L}_\infty} + \lambda^{kN}\|\mathbf{x}(0)\|_{e_\lambda}))\\ &\le \|y_{kN}\|_{\mathcal{L}_\infty}(r_1 + r_2\epsilon) + \|\mathbf{x}(0)\|_{e_\lambda}\lambda^N r_2(1+\epsilon),\end{aligned}$$

where we use (6.3) along with the inequality $\|.\|_{e_1} \le \|.\|_{e_\lambda}$, and r_1 and r_2 are defined in (B.3); therefore

$$\|y_{kN}\|_{\mathcal{L}_\infty} \le (r_1 + r_2\epsilon)^k\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda} r_2(1+\epsilon)\sum_{i=1}^{k}(r_1 + r_2\epsilon)^{k-i}(\lambda^N)^{i-1},$$

for all $k \in \mathbb{N}$. Now take any

$$\epsilon \in \left(\max\left\{0, \frac{1 - r_1 - 2r_2}{3r_2}\right\}, \frac{1-r_1}{r_2}\right),$$

so $r_1 + r_2\epsilon < 1$. Then, letting $\underline{\lambda}_N \triangleq \max\{r_1 + r_2\epsilon, \lambda^N\}$, as before in the linear case of Section 6.4.3, we can find continuous increasing functions

$$\begin{aligned}K_N(\lambda^N) &\triangleq \max\left\{1, \frac{2r_2(1+\epsilon)}{1-\underline{\lambda}_N}\right\} = \frac{2r_2(1+\epsilon)}{1-\underline{\lambda}_N},\\ \gamma_N(\lambda^N) &\triangleq \frac{1+\underline{\lambda}_N}{2},\end{aligned}$$

by Claim 6.2, such that $\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda} < \delta_3$ implies

$$\|y_{kN}\|_{\mathcal{L}_\infty} \le K_N(\lambda^N)\gamma_N(\lambda^N)^k(\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda}),$$

and since $\delta_3 \leq \delta_{\text{fh}}$, by Lemma 6.3

$$\|y_k\|_{\mathcal{L}_\infty} \leq L_{\text{fh}} K_N(\lambda^N) \gamma_N(\lambda^N)^{\bar{k}} (\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda}) + L_{\text{fh}} (\lambda^N)^{\bar{k}} \|\mathbf{x}(0)\|_{e_\lambda},$$

for all $k \in \mathbb{N}$ as $L_{\text{fh}} \geq 1$, where $\bar{k} \in \mathbb{N}$ satisfies $k = \bar{k}N + j$ and $j \in \{0, 1, \ldots, N-1\}$. In turn, this means that

$$\|y_k\|_{\mathcal{L}_\infty} \leq 2L_{\text{fh}} K_N(\lambda^N) \gamma_N(\lambda^N)^{\bar{k}} (\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda}),$$

for all $k \in \mathbb{N}$. Let $\gamma(\lambda) \triangleq (\gamma_N(\lambda^N))^{1/N}$. Then,

$$\|y_k\|_{\mathcal{L}_\infty} \leq 2L_{\text{fh}} K_N(\lambda^N) \gamma(\lambda)^{k-j} (\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda}),$$

hence, as $\gamma(\lambda) \in (0,1)$ and $j \leq N-1$,

$$\|y_k\|_{\mathcal{L}_\infty} \leq \overbrace{2L_{\text{fh}} K_N(\lambda^N) \gamma(\lambda)^{1-N}}^{K(\lambda)} \gamma(\lambda)^k (\|y_0\|_{\mathcal{L}_\infty} + \|\mathbf{x}(0)\|_{e_\lambda}), \tag{B.6}$$

for all $k \in \mathbb{N}$. Clearly, γ is continuous and increasing as before, while K defined in (B.6) is continuous. It remains to show that K is increasing. Noting that

$$\begin{aligned} K(\lambda) &= 2L_{\text{fh}} \frac{K_N(\lambda^N)}{\gamma(\lambda)^N} \gamma(\lambda) = 2L_{\text{fh}} \frac{K_N(\lambda^N)}{\gamma_N(\lambda^N)} \gamma(\lambda) \\ &= 2L_{\text{fh}} \frac{2r_2(1+\epsilon)}{1-\underline{\lambda}_N} \frac{2}{1+\underline{\lambda}_N} \gamma(\lambda) \\ &= 8L_{\text{fh}} r_2 (1+\epsilon) \frac{\gamma(\lambda)}{1-\underline{\lambda}_N^2}, \end{aligned}$$

we can conclude K is also increasing, since $(1-\underline{\lambda}_N^2)^{-1}$ is increasing on $\mathbb{R}^+$ with respect to $\underline{\lambda}_N$.

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